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On the Independence Number of the Generalized Petersen Graph $P(n,k)^*$

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Abstract Let $G=(V(G),E(G))$ be a simple finite undirected graph. A set $S\subseteq V(G)$ is an independent set if no two vertices of S are adjacent. The independence number $\alpha(G)$ is the maximum cardinality of an independent set in G. In this paper, we investigate the independence number of generalized Petersen graph, and give the exact values of $P(n,k)$ for $k=1,2,3,5$.

Keywords Operations Research; Graph Theory; Independent set; Independence number; Generalized Petersen graph

1 Introduction

We only consider simple finite undirected graphs, and use [1] for the terminology and notation not defined in this paper.

A graph $G=(V(G),E(G))$ is a set $V(G)$ of vertices and a subset $E(G)$ of the unordered pairs of vertices, called edges. An induced subgraph $\langle G,S\rangle$ is a subgraph on vertex set $S\subseteq V(G)$ obtained by taking S and all edges of G having both endpoints in S.

The open neighborhood and the closed neighborhood of a vertex $v\in V(G)$ are denoted by $N(v)=\{u\in V(G): vu\in E(G)\}$ and $N[v]=N(v)\cup\{v\}$, respectively. For a vertex set $S\subseteq V(G), N(S)=\bigcup_{v\in S}N(v)$ and $N[S]=\bigcup_{v\in S}N[v]$.

A set $S\subseteq V(G)$ is an independent set if no two vertices of S are adjacent. The independence number $\alpha(G)$ is the maximum cardinality of an independent set in G.

It is the most important parameters in graph theory on which people have concentrated tremendous efforts. There are many results concerning independence of simple graphs. The basic result of upper and/or lower bounds was proved, independently, in [2, 3, 4, 5, 6, 7, 8, 9, 10].

The generalized Petersen graph $P(n,k)$[1, 11, 12] is the graph with vertices $\{v_i,\ u_i : 0\le i\le n-1\}$ and edges $\{v_iv_{i+1},\ v_iu_i,\ u_iu_{i+k}\}$, where subscripts modulo n and $k<n/2$.

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In this paper, we investigate the independence number of generalized Petersen graphs $P(n,k)$ and show that, (1) for odd k and even n, $\alpha(P(n,k)) = n$, (2) for odd n, $\alpha(P(n,1)) = n-1$, $\alpha(P(n,3)) = n-2$, $\alpha(P(n,5)) = n-3$, and (3) $\alpha(P(n,2)) = \lfloor 4n/5 \rfloor$.

2 The independence number of $P(n,k)$

Lemma 2.1.
(1) $\alpha(P(n,2h-1)) \geq n$ for even n,
(2) $\alpha(P(n,2h-1)) \geq n-h$ for odd n,
(3) $\alpha(P(n,2)) \geq \lfloor 4n/5 \rfloor$.

Proof. (1) For even n, let $S = \{v_{2i},\, u_{2i+1} : 0 \leq i \leq n/2-1\}$. Then, S is an independent set of $P(n,2h-1)$ with $|S| = n$. Hence, we have $\alpha(P(n,2h-1)) \geq n$ for even n.

(2) For odd n, let $S = \{v_{2i} : 0 \leq i \leq (n-1)/2-1\} \cup \{u_{2i+1} : 0 \leq i \leq (n-1)/2-h\}$. Then, S is an independent set of $P(n,2h-1)$ with $|S| = (n-1)/2 + (n-1)/2 - h + 1 = n-h$. Hence, we have $\alpha(P(n,2h-1)) \geq n-h$ for odd n.

(3) Let $n = 5m+t$, and

$$S = \begin{cases} \{v_{5i}, v_{5i+3}, u_{5i+1}, u_{5i+2} : 0 \leq i \leq m-1\}, & t = 0,1; \\ \{v_{5i}, v_{5i+3}, u_{5i+1}, u_{5i+2} : 0 \leq i \leq m-1\} \cup \{v_{5m}\}, & t = 2; \\ \{v_{5i}, v_{5i+3}, u_{5i+1}, u_{5i+2} : 0 \leq i \leq m-1\} \cup \{v_{5m}, u_{5m+1}\}, & t = 3; \\ \{v_{5i}, v_{5i+3}, u_{5i+1}, u_{5i+2} : 0 \leq i \leq m-1\} \cup \{v_{5m}, u_{5m+1}, u_{5m+2}\}, & t = 4. \end{cases}$$

Then, S is an independent set of $P(n,2)$ with $|S| = \lfloor 4n/5 \rfloor$, we have $\alpha(P(n,2)) \geq \lfloor 4n/5 \rfloor$. □

In Figure 2.1, we show some independent sets of $P(n,k)$ in Lemma 2.1, where the vertices of S are in dark.

Theorem 2.2. $\alpha(P(n,2h-1)) = n$ for even n.

Proof. Let S be an arbitrary independent set of $P(n,k)$, then $|S| = \sum_{i=0}^{n-1} |S \cap \{v_i, u_i\}| \leq \sum_{i=0}^{n-1} 1 = n$. Hence, $\alpha(P(n,k)) \leq n$, it follows that $\alpha(P(n,2h-1)) \leq n$. By Lemma 2.1(1), $\alpha(P(n,2h-1)) \geq n$, we have $\alpha(P(n,2h-1)) = n$ for even n.

□

For convenience, let $V_v = \{v_i : 0 \leq i \leq n-1\}$ and $V_u = \{u_i : 0 \leq i \leq n-1\}$, then $V(P(n,k)) = V_v \cup V_u$.

Theorem 2.3. $\alpha(P(n,1)) = n-1$ for odd n.

Proof. For odd n, let S be an arbitrary independent set of $P(n,1)$, then $|S \cap V_v| \leq (n-1)/2$ and $|S \cap V_u| \leq (n-1)/2$. Hence, $|S| \leq (n-1)/2 + (n-1)/2 = n-1$, it follows that $\alpha(P(n,1)) \leq n-1$. By lemma 2.1(2), $\alpha(P(n,1)) \geq n-1$, we have $\alpha(P(n,1)) = n-1$ for odd n. □

Theorem 2.4. $\alpha(P(n,3)) = n-2$ for odd n.

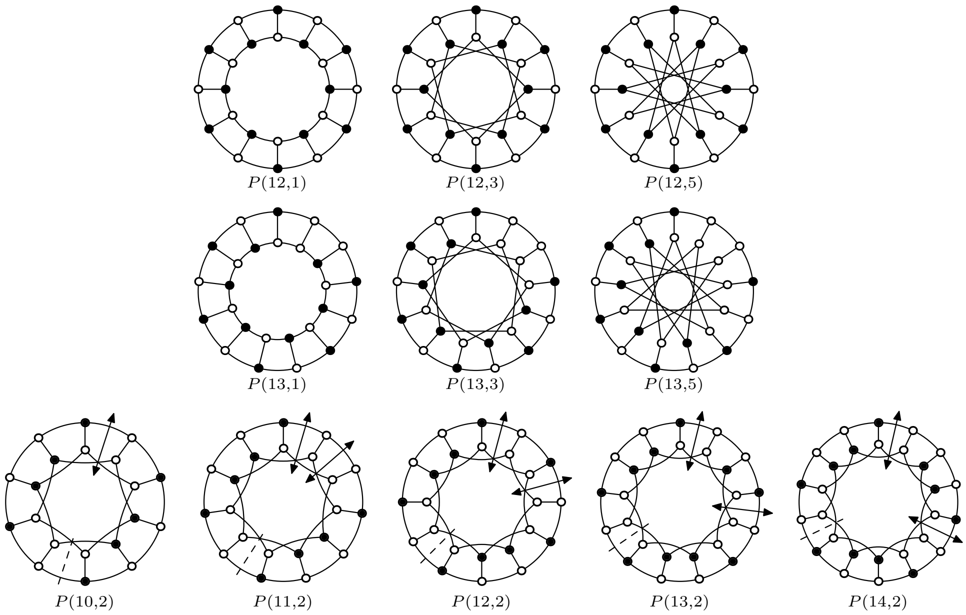

Figure 2.1: Some independent sets of $P(n,k)$

Proof. Let S be an arbitrary independent set of $P(n,3)$. For odd n, let $n = 2m+1$, $|S \cap V_v| = x$ and $|S \cap V_u| = y$. Then, $\alpha(P(n,3)) = x+y$ and $x \leq m$, $y \leq m$.

If $x \leq m-1$, then $x+y \leq m-1+m = 2m-1 = n-2$.

If $x = m$, then, without loss of generality, we may assume that $S \cap V_v = \{v_{2i} : 0 \leq i \leq m-1\}$(see Figure 2.2, where the vertices of $S \cap V_v$ are in dark and the vertices of $V_u \cap N[S \cap V_v]$ are marked by slash). It follows that $S \cap V_u \subseteq \{u_{2i+1} : 0 \leq i \leq m-1\} \cup \{u_{2m}\}$ with $|S \cap V_u| \leq m+1$. Since $u_{2m-3}u_{2m}$, $u_{2m-1}u_1 \in E(P(n,3))$, we have $|S \cap \{u_{2m-3}, u_{2m-1}, u_{2m}, u_1 \in S_u\}| \leq 2$, it follows $y \leq m+1-2 = m-1$ and $x+y \leq m+m-1 = 2m-1 = n-2$.

Hence, $\alpha(P(n,3)) \leq n-2$. By Lemma 2.1(2), $\alpha(P(n,3)) \geq n-2$, we have $\alpha(P(n,3)) = n-2$ for odd n.

□

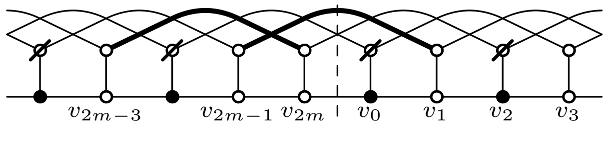

Figure 2.2: An independent set S of $P(n,3)$ with $|S \cap V_v| = m$

Theorem 2.5. $\alpha(P(n,5)) = n-3$ for odd n.

Proof. We left reader to verify that $\alpha(P(11,5)) = 8$. For odd $n \geq 13$, let S be an arbitrary independent set of $P(n,5)$. Let $n = 2m+1$, $|S \cap V_v| = x$ and $|S \cap V_u| = y$. Then, $\alpha(P(n,5)) = x+y$ and $x \leq m$, $y \leq m$.

Case 1. $x \leq m-2$. Then $x+y \leq m-2+m = 2m-2 = n-3$.

Case 2. $x = m-1$. Then, every connected component of $\langle P(n,5), V_v - S\rangle$ is a path of length at most three. Let z_i be the number of paths with length i, then $z_0+z_1+z_2+z_3 = m-1$ and $z_0+2z_1+3z_2+4z_3 = m+2$. Hence, $z_1+2z_2+3z_3 = 3$. There are three subcases:

Case 2.1. $z_3 = 1$. Then $z_1 = z_2 = 0$ and $z_0 = m-2$. Without loss of generality, we may assume that $S \cap V_v = \{v_0\} \cup \{v_{2i+1} : 2 \leq i \leq m-1\}$(see Figure 2.3(a)). It follows $S \cap V_u \subseteq \{u_{2i} : 1 \leq i \leq m\} \cup \{u_1, u_3\}$ and $y = |S \cap V_u| \leq m+2$. Since $u_{2m}u_4$, u_1u_6, $u_3u_8 \in E(P(n,5))$, we have $|S \cap \{u_{2m}, u_1, u_3, u_4, u_6, u_8\}| \leq 3$, it follows $y \leq m+2-3 = m-1$.

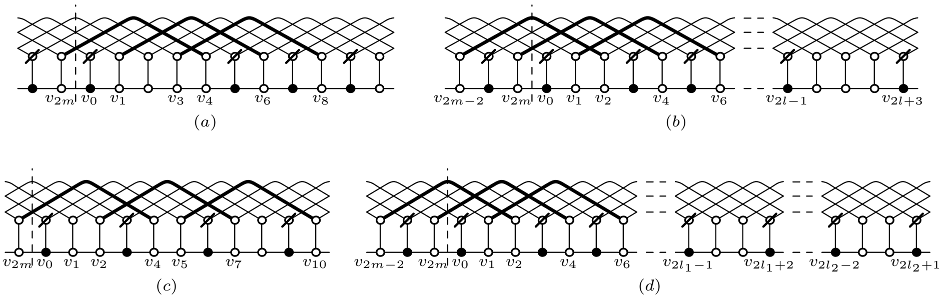

Figure 2.3: Some independent sets S of $P(n,5)$ with $|S \cap V_v| = m-1$

Case 2.2. $z_3 = 0$ and $z_2 = 1$. Then $z_1 = 1$ and $z_0 = m-3$. Without loss of generality, we may assume that $S \cap V_v = \{v_0, v_3\} \cup \{v_{2i+1} : 2 \leq i \leq l-1\} \cup \{v_{2i+1} : l+1 \leq i \leq m-1\}$ where $2 \leq l \leq m-1$(see Figure 2.3(b)). It follows that $S \cap V_u \subseteq \{u_{2i} : 1 \leq i \leq m\} \cup \{u_1, u_{2l+1}\}$ and $y = |S \cap V_u| \leq m+2$. Since $u_{2m-2}u_2$, $u_{2m}u_4$, $u_1u_6 \in E(P(n,5))$, we have $|S \cap \{u_{2m-2}, u_{2m}, u_1, u_2, u_4, u_6\}| \leq 3$, it follows $y \leq m+2-3 = m-1$.

Case 2.3. $z_3 = 0$ and $z_2 = 0$. Then $z_1 = 3$ and $z_0 = m-4$. We denote these paths with length 1 as P_1^1, P_1^2 and P_1^3. By symmetry, we only need to consider two subcases:

Case 2.3.1. $N[P_1^1] \cap N[P_1^2] \neq \emptyset$ and $N[P_1^2] \cap N[P_1^3] \neq \emptyset$. Without loss of generality, we may assume that $S \cap V_v = \{v_0, v_3, v_6, v_9\} \cup \{v_{2i+1} : 5 \leq i \leq m-1\}$(see Figure 2.3(c)). It follows that $S \cap V_u \subseteq \{u_{2i} : 5 \leq i \leq m\} \cup \{u_1, u_2, u_4, u_5, u_7, u_8\}$ and $y = |S \cap V_u| \leq m+2$. Since $u_{2m}u_4$, u_2u_7, $u_5u_{10} \in E(P(n,5))$, we have $|S \cap \{u_{2m}, u_2, u_4, u_5, u_7, u_{10}\}| \leq 3$, it follows $y \leq m+2-3 = m-1$.

Case 2.3.2. $N[P_1^1] \cap N[P_1^2] = \emptyset$ and $N[P_1^2] \cap N[P_1^3] = \emptyset$. Without loss of generality, we may assume that $S \cap V_v = \{v_0, v_3, v_5\} \cup \{v_{2i+1} : 3 \leq i \leq l_1-1\} \cup \{v_{2i} : l_1+1 \leq i \leq l_2-1\} \cup \{v_{2i+1} : l_2 \leq i \leq m-1\}$ where $3 \leq l_1 < l_2 \leq m-1$(see Figure 2.3(d)). It follows that $S \cap V_u \subseteq \{u_{2i} : 1 \leq i \leq l_1\} \cup \{u_{2i+1} : l_1 \leq i \leq l_2-1\} \cup \{u_{2i} : l_2 \leq i \leq m\} \cup \{u_1\}$ and $y = |S \cap V_u| \leq m+2$. Since $u_{2m-2}u_2$, $u_{2m}u_4$, $u_1u_6 \in E(P(n,5))$, we have $|S \cap \{u_{2m-2}, u_{2m}, u_1, u_2, u_4, u_6\}| \leq 3$, it follows $y \leq m+2-3 = m-1$.

From subcases 2.1-2.3, we have $x+y \leq m-1+m-1 = 2m-2 = n-3$ for $x = m-1$.

Case 3. $x = m$. Then, without loss of generality, we may assume that $S \cap V_v = \{v_{2i} : 0 \le i \le m-1\}$(see Figure 2.4). It follows that $S \cap V_u \subseteq \{u_{2i+1} : 0 \le i \le m-1\} \cup \{u_{2m}\}$ and $|S \cap V_u| \le m+1$. Since $u_{2m-5}u_{2m}$, $u_{2m-3}u_1$, $u_{2m-1}u_3 \in E(P(n,5))$, we have $|S \cap \{u_{2m-5}, u_{2m}, u_{2m-3}, u_{2m-1}, u_1, u_3\}| \le 3$, it follows $y \le m+1-3 = m-2$ and $x+y \le m+m-2 = 2m-2 = n-3$.

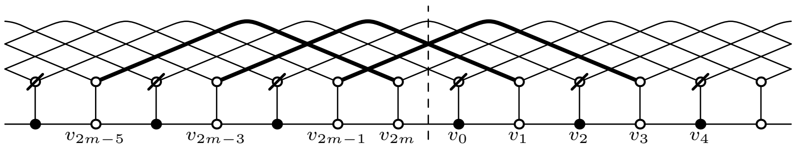

Figure 2.4: An independent set S of $P(n,5)$ with $|S \cap V_v| = m$

From cases 1-3, we have $\alpha(P(n,5)) \le n-3$.

By Lemma 2.1(2), $\alpha(P(n,5)) \ge n-3$, we have $\alpha(P(n,5)) = n-3$ for odd n.

□

Let $V'(i,l) = \{v_{i+w}, u_{i+w} : 0 \le w \le l-1\} \subseteq V(P(n,2))$.

Lemma 2.6. Let S be an arbitrary independent set of $P(n,2)$, then $|S \cap V'(i,5)| \le 4$.

Proof. Since $u_{i+1}u_{i+3} \in E(P(n,2))$, $|S \cap \{u_{i+1}, u_{i+3}\}| \le 1$.

Case 1. $|S \cap \{u_{i+1}, u_{i+3}\}| = 0$. Then, $|S \cap V'(i,5)| = |S \cap \{u_{i+1}, u_{i+3}\}| + |S \cap \{u_i, v_i, v_{i+1}\}| + |S \cap \{u_{i+2}, v_{i+2}, v_{i+3}, v_{i+4}, u_{i+4}\}| \le 0+2+2 = 4$ (see Figure 2.5(a)).

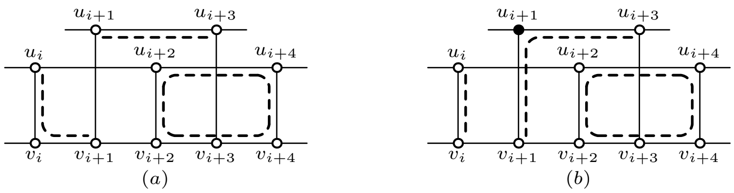

Figure 2.5: $|S \cap V'(i,5)| \le 4$

Case 2. $|S \cap \{u_{i+1}, u_{i+3}\}| = 1$, say $u_{i+1} \in S$. Then $|S \cap V'(i,5)| = |S \cap \{u_{i+1}, u_{i+3}, v_{i+1}\}| + |S \cap \{u_i, v_i\}| + |S \cap \{u_{i+2}, v_{i+2}, v_{i+3}\, v_{i+4}, u_{i+4}\}| \le 1+1+2 = 4$ (see Figure 2.5(b)).

From Cases 1-2, we have $|S \cap V'(i,5)| \le 4$.

□

Theorem 2.7. $\alpha(P(n,2)) = \lfloor 4n/5 \rfloor$.

Proof. Let S be an arbitrary independent set of $P(n,2)$.

Case 1. $n \equiv 0(mod\ 5)$. Then, by Lemma 2.6, we have $|S| = \Sigma_{i=0}^{n/5-1} |S \cap V'(5i,5)| \le (n/5) \times 4 = 4n/5 = \lfloor 4n/5 \rfloor$.

Case 2. $n \not\equiv 0(mod\ 5)$. Then, by Lemma 2.6, we have $5|S| = \Sigma_{i=0}^{n-1} |S \cap V'(5i,5)| \le n \times 4 = 4n$. Hence $|S| \le \lfloor 4n/5 \rfloor$.

From Cases 1-2, we have $\alpha(P(n,2)) \leq \lfloor 4n/5 \rfloor$.

By Lemma 2.1(3), $\alpha(P(n,2)) \geq \lfloor 4n/5 \rfloor$, we have $\alpha(P(n,2)) = \lfloor 4n/5 \rfloor$.

□

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