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Numerical Computation for Demyanov Difference of Polyhedral Convex Sets

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Abstract Under the assumption of polyhedron, the Demyanov difference of convex compact sets is computed via some linear programmings by C procedure. The difference from the existing results is also compared, and some examples with the corresponding numerical results are given.

Keywords Demyanov difference; Polyhedron

1 Introduction

Many practical problems can be described adequately only with the aid of nonsmooth functions and Minkowski duality plays a very important role in nonsmooth analysis and optimization. Take example for convex functions, due to Minkowski duality convex compact subsets corresponding to directional derivatives (which is sublinear) are the subdifferential.

However, if the directional derivative of a nonsmooth function is neither sublinear nor superlinear, then classical Minkowski duality can't be used to establish a relationship between directional derivative and generalized differential. For this, Demyanov introduced a difference of convex compact subsets, which is called Demyanov difference later and is helpful to extend the classical Minkowski duality to a wider class of functions, see for example [1], [2]. Whereafter, many articles on Demyanov difference appeared, which can be mainly divided into two classes,

1. The operational property of Demyanov difference, see for example [1, 2, 6, 7];
2. The application in optimality conditions for quasidifferentiable optimization, see for example [3, 4].

Because the definition of the Demyanov difference is complicated, in practice, it can only be computed for a few examples which affects its usefulness in algorithm. In [3], the computation of Demyanov difference for polyhedron is done by some linear programming. In this paper, under the assumption of polyhedron convex compact sets, the Demyanov difference is also computed by some linear programmings which is the same as [3] in quantity. But the construction of every linear programming is different to the one

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in [3]. The method can be realized in numerically by C procedure or other ones. In this paper, the examples are presented and the corresponding results are given by C procedure which is more adaptive than the one in [3] for some cases.

2 Numerical computation for Demyanov difference

Firstly, concept of the Demyanov difference is recalled, which can be found in [2, 6, 7].

Definition 2.1.
Let $U,V \subset R^n$ be convex compact, $P_U(d) = \max_{u\in U}\langle u,d\rangle$, $P_V(d) = \max_{v\in V}\langle v,d\rangle$, $T_{U,V}$ is a set of full measure with respect to R^n in which the functions $P_U(d)$ and $P_V(d)$ are also differentiable, the Demyanov difference of U and V is defined by

$$U \dot{-} V = \text{clco}\{\nabla P_U(d) - \nabla P_V(d) \,|\, d \in T_{U,V}\}.$$

Remark 2.1.
The Demyanov difference is consistent for the different choice of $T_{U,V}$, so the above definition is reasonable.

In order to compute $U \dot{-} V$, it is necessary to know what is $T_{U,V}$ and how to compute $\nabla P_U(d)$ and $\nabla P_V(d)$ in Definition 2.1. By convex analysis [5], one has the function $P_U(d)$ is subdifferentiable and for any $d \in R^n$,

$$\partial P_U(d) = \{u \in U \,|\, \langle u,d\rangle = P_U(d)\} \tag{2.1}$$

Specially, if $\partial P_U(d) = \{\nabla P_U(d)\}$ is singleton, then P is differentiable at d, i.e., $\nabla P_U(d)$ exists. It can be known that $P_U(d)$ is differentiable a.e. by convex analysis [5], so the set $\{d \in R^n \,|\, \nabla P_U(d)\,\text{exsists}\}$ is of full measure with respect to R^n. Then, it is easy to see that the set $\{d \in R^n \,|\, \nabla P_U(d)\,\text{and}\,\nabla P_V(d)\,\text{exsists}\}$ is also of full measure with respect to R^n. By Remark 2.1, $T_{U,V}$ can be understood by the following way,

$$T_{U,V} = \{d \in R^n \,|\, \nabla P_U(d)\,\text{and}\,\nabla P_V(d)\,\text{exsists}\}.$$

Now, we consider how to compute the sets $\nabla P_U(d)$ and $\nabla P_V(d)$. To be more intuitive, assume U and V are polyhedron. We first discuss $\nabla P_U(d)$ for the polyhedron U in the following Figure 1.

1. If $\nabla P_U(d)$ exists for some d, $\nabla P_U(d)$ must be some vertex of the polyhedron U by (2.1).
2. If u_1 is the gradient at some d, then u_1 is the only point in the set $\{u \in R^n \,|\, \langle u,d\rangle = P_U(d)\}$. It intuitively equal to $d \in R^n \backslash \{0_n\}$ and the following inequalities hold:

   $$(u_i - u_1)^T d < 0, \quad i = 2,3,4,5$$

   That is to say for all $d \in R^n$ satisfying $(u_i - u_1)^T d < 0, i = 2,3,4,5$, in the figure 1 , $\nabla P_U(d) = u_1$ holds.

By the above analysis, if $V = \text{co}\{v_1, v_2, v_3, v_4\}$, then $u_1 - v_1$ can be expressed by $\nabla P_U(d) - \nabla P_V(d)$ for some $d \in T_{U,V}$ iff the following inequalities for d has at least one nonzero solution:

$$\begin{pmatrix} (u_i - u_1)^T \\ (v_j - v_1)^T \end{pmatrix} d < 0, \quad i = 2,3,4,5, j = 2,3,4 \tag{2.2}$$

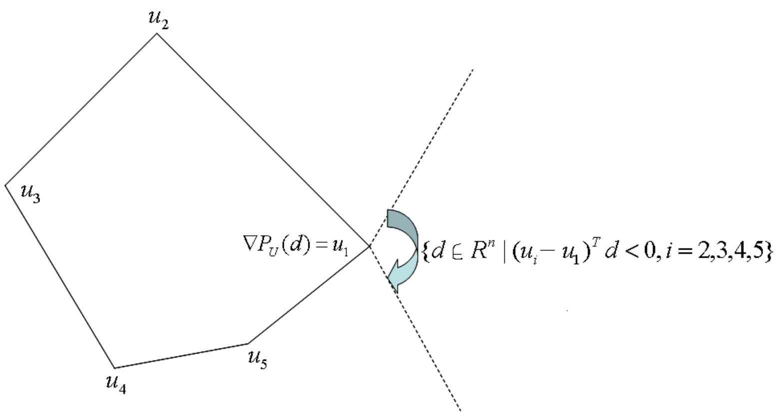

Figure 1: $\nabla P_U(d)$ for the polyhedron U

Naturally, the following results is reasonable,

Lemma 2.1.
Suppose $I = \{1, \cdots, p\}$, $J = \{1, \cdots, q\}$, $U = \text{co}\{u_i \mid i \in I\}$, $V = \text{co}\{v_j \mid j \in J\}$, then

$$U \dot{-} V = \text{co}\left\{ u_s - v_t \;\middle|\; \exists d \in R^n : \quad \begin{pmatrix} (u_i - u_s)^T \\ (v_j - v_t)^T \end{pmatrix} d < 0, \quad i \in I \backslash \{s\}, \, j \in J \backslash \{t\} \right\} \tag{2.3}$$

The proof of the above result can be referred to Y. Gao in [3].

By far, the computation of Demyanov difference has been changed to the computation of the inequalities. To solve the above inequalities, the following work is done.

Lemma 2.2.
Inequalities for d, $Ad < 0$, have no nonzero solution iff $0_n \in \text{co}\{a_i \mid i = 1, \cdots, m\}$, where $A = (a_1^T, \cdots, a_m^T)^T$, $a_i \in R^n$, $d \in R^n$.

Proof Inequalities for d, $Ad < 0$, means $\{d \mid a_i^T d < 0, i = 1, \cdots, m\}$. Firstly, consider the **necessaries**. Assume by contradiction $0_n \notin \text{co}\{a_i \mid i = 1, \cdots, m\}$, for $\text{co}\{a_i \mid i = 1, \cdots, m\}$ is a polyhedron, there exists a nonzero vector $d \in R^n$ such that $\max\{\langle a_i, d\rangle \mid i = 1, \cdots, m\} < 0$ by seperated theorem [5]. That is to say the nonzero vector d satisfies $Ad < 0$, which is a contradition. Thus $0_n \in \text{co}\{a_i \mid i = 1, \cdots, m\}$.

Sufficiency. Assume by contadiction that there exists a nonzero vector $d \in R^n$ such that $a_i^T d < 0, i = 1, \cdots, m$. Since $0_n \in \text{co}\{a_i \mid i = 1, \cdots, m\}$, there exists a set of nonzero numbers $r_i \geq 0, i = 1, \cdots, m$, $\sum_{i=1}^m r_i = 1$, such that $0_n = \sum_{i=1}^m r_i a_i$ by the definition of convex hull. So

$$0 = 0_n^T d = (\sum_{i=1}^m r_i a_i)^T d = \sum_{i=1}^m (r_i a_i^T) d = \sum_{i=1}^m r_i a_i^T d < 0,$$

which is impossible. Therefore $Ad < 0$ have no nonzero solution. □

By Lemma 2.2, in order to tell whether the inequalities for d, $Ad < 0$, have a nonzero solution, it is sufficient to consider whether the set $\mathrm{co}\{a_i \,|\, i = 1, \cdots, m\}$ contains 0_n. The following transfer will be done which makes it executive to tell whether $0_n \notin \mathrm{co}\{a_i \,|\, i = 1, \cdots, m\}$.

Theorem 2.1.
Let $a_i \in R^n, i = 1, \cdots, m$. Then $0_n \notin \mathrm{co}\{a_i \,|\, i = 1, \cdots, m\}$ iff the optimal value of the following linear programming is 0,

$$\begin{array}{ll} \min & -\sum_{i=1}^m r_i \\ \text{s.t.} & \sum_{i=1}^m r_i a_i = 0_n, \\ & 0 \le r_i \le 1, \qquad i = 1, \cdots, m \end{array} \tag{2.4}$$

Proof Necessity. Since $0_n \notin \mathrm{co}\{a_i \,|\, i = 1, \cdots, m\}$, the set of number $r_i \ge 0$, $i = 1, \cdots, m$, such that $\sum_{i=1}^m r_i a_i = 0_n$, are only zeroes. That is to say the feasible region of the linear programming (2.4) contains 0_n only. So the optimal value of the problem (2.4) is 0.

Sufficiency. By the fact that the optimal value of the problem (2.4) is zero, it can be deduced that the scalars satisfying $\sum_{i=1}^m r_i a_i = 0_n$, $0 \le r_i \le 1$, is the only 0_n. From the definition of the convex hull, $0_n \notin \mathrm{co}\{a_i \,|\, i = 1, \cdots, m\}$. □

Theorem 2.2.
Let $U = \mathrm{co}\{u_i \,|\, i = 1, \cdots, p\} \subseteq R^n$, $V = \mathrm{co}\{v_j \,|\, j = 1, \cdots, q\} \subseteq R^n$, $s \in I := \{1, \cdots, p\}$, $t \in J := \{1, \cdots, q\}$. Then $u_s - v_t \in U \dot{-} V$ iff 0 is the optimal value of the following problem

$$\begin{array}{ll} \min & -\sum_{i \in (I \cup J) \setminus \{s,t\}} r_i \\ \text{s.t.} & \sum_{i \in I \setminus \{s\}} r_i (u_i - u_s) = 0_n, \\ & \sum_{j \in J \setminus \{t\}} r_j (v_j - v_t) = 0_n, \\ & 0 \le r_i \le 1, \quad i \in (I \cup J) \setminus \{s,t\} \end{array} \tag{2.5}$$

Proof By Lemma 2.1, $u_s - v_t \in U \dot{-} V$ iff the inequalities for d

$$\begin{pmatrix} (u_i - u_s)^T \\ (v_j - v_t)^T \end{pmatrix} d < 0, \quad i \in I \setminus \{s\}, j \in J \setminus \{t\} \tag{2.6}$$

has at least one nonzero solution. By Lemma 2.2, that is to say

$$0_n \notin \mathrm{co}\{u_i - u_s, v_j - v_t \,|\, i \in I \setminus \{s\}, j \in J \setminus \{t\}\}.$$

Applying Theorem 2.1, one has the optimal value for linear programming (2.5) is 0, which completes the proof of the theorem. □

In [3], the Demyanov difference of polyhedral convex sets is determined by finitely number of linear programming problems which are constructed by Gorden Theorem. In the construction, every linear programming problem is $n + I + J - 2$ dimensional, and the constraints are composed of $n + 1$ equality ones, where n is the dimension of polyhedral convex sets, I and J are the numeral of vertex of the two polyhedral convex sets in Demyanov difference, respectively. In this paper, the computation of the Demyanov difference is also determined by finitely number of linear programming problems, and the number of linear programming problems is the same to Gao in [3]. However, the construction of every linear programming problem is different from Gao in [3], the dimension of the variable is $I+J-2$, and the number of the equality constraints is $I+J-2$, in which n, I and J are of the same meaning to above. Obviously, the dimensional is decreased with respect to [3], and the number of the constraints is neither better nor worse than [3], which will be different to different problems. And if the dimension of the polyhedral convex sets is relative large and the vertex of the polyhedron is not big, the method in this paper will be more adaptive than [3].

3 Algorithm and Numerical Results

Basing on the results in Section 2, the executive algorithm for Demyanov difference of two convex compact sets can be given as follows under the assumption that the convex compact sets are polyhedron.

Algorithm 3.1.
(Compute $U \dot{-} V$, where $U = \mathrm{co}\{U_i \,|\, i = 1, \cdots, p\}$, and $V = \mathrm{co}\{V_j \,|\, j = 1, \cdots, q\}$)

Step0 *Initiation U_i, V_j, $i := 1, j := 1$.*
Setp1 *Compute $a_{ij} = U_i - V_j$, construct the following programming*

$$\begin{array}{ll} \min & -\sum_{k=1}^{m} r_k \\ \text{s.t.} & \sum_{k=1}^{m} r_k a_{ij} = 0_n, \\ & 0 \le r_k \le 1, \qquad k = 1, \cdots, m \end{array} \tag{3.7}$$

If the optimal value of (3.7) is 0, output a_{ij}. Otherwise go to Step 2.
Step2 *If $j < q$, set $j = j+1$, go to Step 1. Otherwise go to Step 3.*
Step3 *If $i < p$, set $i = i+1$, go to Step 1. Otherwise Stop.*

What follows, we shall choose some polyhedron randomly and give the numerical results basing on the algorithm given above which is written by C procedure.

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Table 1: Demyanov difference of polyhedron

Problem 1	$U = \text{co}\{u_1, u_2, u_3\}$ $u_1 = (1, -3.2, 5, 6.1)$ $u_2 = (3.3, 3.5, 1.7, 5)$ $u_3 = (6, 6.9, 6.1, 2.1)$ $V = \text{co}\{v_1, v_2, v_3, v_4\}$ $v_1 = (1.2, 2.1, 3, 5)$ $v_2 = (22, 10, 1.5, 0)$ $v_3 = (-1.01, 0, 5.53, 26)$ $v_4 = (-3, -4, 0.5, 6)$	$U \dot{-} V = \text{co}\{a_{ij}, \lvert i = 1, \cdots, 3, j = 1, \cdots, 4\}$ $a_{11} = (-0.2, -5.3, 2, 1.1)$ $a_{12} = (-21, -13.2, 3.5, 6.1)$ $a_{13} = (2.01, -3.2, -0.53, -19.9)$ $a_{14} = (4, 0.8, 4.5, 0.1)$ $a_{21} = (2.1, 1.4, -1.3, 0)$ $a_{22} = (-18.7, -6.5, 0.2, 5)$ $a_{23} = (4.31, 3.5, -3.83, -21)$ $a_{24} = (6.3, 7.5, 1.2, -1)$ $a_{31} = (4.8, 4.8, 3.1, -2.9)$ $a_{32} = (-16, -3.1, 4.6, 2.1)$ $a_{33} = (7.01, 6.9, 0.57, -23.9)$ $a_{34} = (9, 10.9, 5.6, -3.9)$
Problem 2	$U = \text{co}\{u_1, u_2\}$ $u_1 = (1, 1, 3.01, 25, 7)$ $u_2 = (29.1, 25.33, 0, -8, -1.1)$ $V = \text{co}\{v_1, v_2, v_3, v_4, v_5\}$ $v_1 = (0.01, 12, 32, 0, -1.52)$ $v_2 = (-100, 6.02, 25, -10, 4)$ $v_3 = (7, 76, 54, -21, -3.3)$ $v_4 = (-1, -1, -1, -1, -1)$ $v_5 = (2, 0, 0.1, -2.1, 1)$	$U \dot{-} V = \text{co}\{a_{ij}, \lvert i = 1, \cdots, 2, j = 1, \cdots, 5\}$ $a_{11} = (0.99, -11, -28.99, 25, 8.52)$ $a_{12} = (101, -5.02, -21.99, 35, 3)$ $a_{13} = (-6, -75, -50.99, 46, 10.3)$ $a_{14} = (2, 2, 4.01, 26, 8)$ $a_{15} = (-1, 1, 2.91, 27.1, 6)$ $a_{21} = (29.09, 13.33, -32, -8, 0.42)$ $a_{22} = (129.1, 19.31, -25, 2, -5.1)$ $a_{23} = (22.1, -50.67, -54, 13, 2.2)$ $a_{24} = (30.1, 26.33, 1, -7, -0.1)$ $a_{25} = (27.1, 25.33, -0.1, -5.9, -2.1)$

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