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Total Coloring of Planar Graphs without Adjacent 4-cycles

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Abstract Let G be a planar graph with maximum degree Δ. It's proved that if $\Delta \geq 8$ and G does not contain adjacent 4-cycles, then the total chromatic number $\chi^{''}(G) = \Delta + 1$.

Keywords planar graph; total coloring; adjacent cycle

1 Introduction

In this paper, all graphs are finite, simple and undirected. Any undefined notation follows that of Bondy and Murty [2]. Let G be a graph, We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply V, E, Δ and δ) to denote the vertex set, the edge set, the maximum(vertex) degree and the minimum (vertex) degree of G, respectively. A k-, k^+- or k^-- vertex is a vertex of degree k, at least k, or at most k, respectively.

A *total-k-coloring* of a graph G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi^{''}(G)$ of G is the smallest integer k such that G has a total-k-coloring. It's clear that $\chi^{''}(G) \geq \Delta + 1$. Behzad [1] and Vizing [13] conjectured that $\chi^{''}(G) \leq \Delta + 2$ for each graph G. This conjecture was verified by Rosenfeld [9] and Vijayaditya [12] for $\Delta = 3$ and by Kostochka [8] for $\Delta \leq 5$. In 1989, Sánchez-Arroyo [11] proved that deciding whether $\chi^{''}(G) = \Delta + 1$ is NP-complete. But For planar graphs with large maximum degree, it is possible to determine $\chi^{''}(G)$ precisely. It is shown that $\chi^{''}(G) = \Delta + 1$ if G is a planar graph with $\Delta \geq 11$ [3] and $\Delta = 10$ [14] and $\Delta = 9$ [7]. Borodin et al. [4] also obtained several related results by adding girth restrictions. Hou et al. [6] proved that if G is a planar graph with $\Delta \geq 8$ and without i-cycles for some $i \in \{5, 6\}$, then $\chi^{''}(G) = \Delta + 1$. Recently D.Z. Du, L. Shen and Y.Q. Wang [5] also proved that if G is a planar graph with $\Delta \geq 8$ and without adjacent 3-cycles, then $\chi^{''}(G) = \Delta + 1$. In this paper, we get the following theorem.

Theorem 1. Let G be a planar graph with $\Delta(G) \geq 8$. If G does not contain adjacent 4-cycles, then $\chi^{''}(G) = \Delta + 1$.

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Let us introduce some notations and definitions. Let $G = (V,E,F)$ be a planar graph, where F is the set of faces of G. The degree of f, denoted by $d(f)$, is the number of edges incident with it. A k-, k^+- or k^--face is a face of degree k, at least k or at most k, respectively. Let $\delta(f)$ denote the minimum degree of vertices incident with f. We say that two cycles are adjacent if they share at least one edge. For $v \in V(G)$, we use $n_i(v)$ to denote the number of i-vertices which are adjacent to v, $f_i(v)$ to denote the number of i-faces incident with v. The vertex marked by $\bullet$ denotes it has no other neighbors in G.

2 Proof of Theorem 1

Proof of Theorem 1. It suffices to consider the case that $\Delta(G) = 8$ by [7]. Let $G = (V,E,F)$ be a minimal counterexample to the theorem in terms of vertices and edges. Then every proper subgraph of G is total-9-colorable. Let L be the color set $\{1,2,\dots,9\}$ for simplicity. It's easy to see that G is 2-connected, and hence has no vertices of degree 1 and the boundary of each face f is exactly a cycle(i.e., $b(f)$ can not pass through a vertex v more than once). First we prove some lemmas for G.

Lemma 1. (a) For any $uv \in E(G)$, $d_G(u) + d_G(v) \geq \Delta + 2 \geq 10$.
(b) The subgraph induced by all (2,6)-edges in G is a forest.

The proof of Lemma 1 can be found in [3].

By Lemma 1, we have that the two neighbors of a 2-vertex are 8-vertices; any two 4^--vertices are not adjacent; any 3-face is incident with three 5^+-vertices, or at least two 6^+-vertices. Let G_2 be the subgraph induced by the edges incident with the 2-vertices of G. In each component T of G_2, if $|V(T)| \geq 4$, then there is a matching M in T which saturating all 2-vertices. If $uv \in M$ and $d(u) = 2$, v is called the *general* 2-*master* of u. Otherwise, T is a path v_1vv_2 where $d(v) = 2$ and v_i is adjacent to exactly one 2-vertex for $i = 1,\ 2$. In this case, the vertex v_i is called the *special* 2-*master* of v for $i = 1,\ 2$.

Lemma 2. G contains no subgraph isomorphic to the configuration in Figure 1(a)-(e).

The proof of (a) and (d) can be found in [10], (b) and (c) can be found in [14], And (e) can be found in [5].

Lemma 3. G contains no subgraph isomorphic to the configuration in Figure 2(a).

proof. On the contrary, suppose G contains the configuration in 2(a). by the minimality of G, $G' = G - uv$ has a proper total-9-coloring φ. For each element $x \in V \cup E$, Let $C(x)$ denotes the set of colors of vertices and edges incident or adjacent to x. Since $|C(v)| \leq 6$ for each 3^--vertex, we suppose that such vertices are colored at the very end. We have $|C(uv)| = 9$, Since otherwise there exists a color $\alpha \in L \setminus C(uv)$, we can color uv

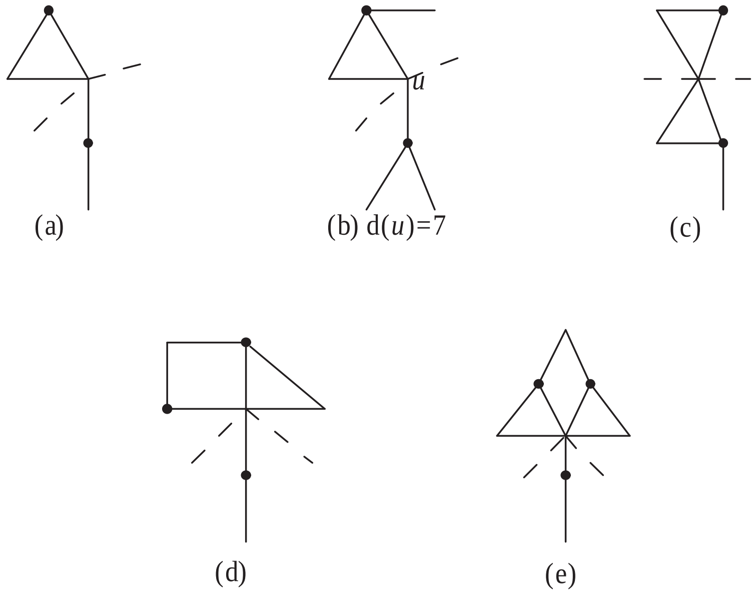


Figure 1 Reducible Configuration

with α to obtain a total-9-coloring of G, a contradiction. Without loss of generality, we can assume that the coloring is one of the Figure 2(b). If $\varphi(wx) \neq 9$, then we can recolor uw with 9, and color uv with $\varphi(uw)$ to obtain a total-9-coloring of G, a contradiction, so $\varphi(wx) = 9$. Similarly, we can prove that $\varphi(zy) = 9$. If $\varphi(xy) \neq 2$, we interchange the colors of the edges wx and xy, and recolor yz with $\varphi(xy)$, recolor uw with 9, color uv with 1. Otherwise, we can interchange the colors of the edges wx and xy, and recolor yz with $\varphi(xy)$, recolor uz with 9, color uv with 2. So we can get total coloring of G with colors from L, a contradiction. Hence we complete the proof of Lemma 3.

Lemma 4. Since G contains no adjacent 4-cycles, then the following results hold:
(a) Any 4^+-vertex is incident with at most $\lfloor \frac{2d(v)}{3} \rfloor$ 3-faces.
(b) Any vertex is incident with at most $\lfloor \frac{d(v)}{2} \rfloor$ 4-faces.

By Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -12 < 0. \tag{1}$$

We define ch to be the initial charge. Let $ch(x) = 2d(x) - 6$ for each $x \in V(G)$ and $ch(x) = d(x) - 6$ for each $x \in F(G)$. In the following, we will reassign a new charge

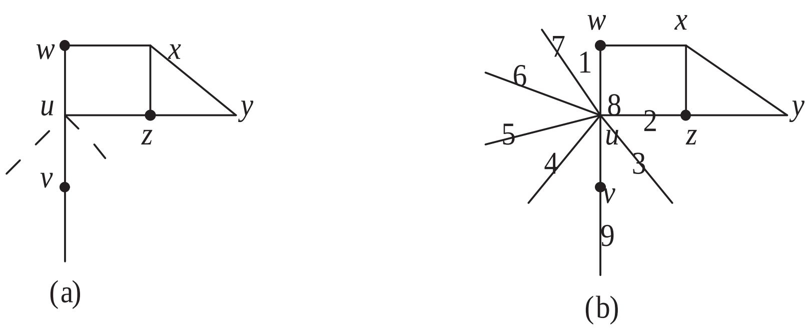


Figure 2 Reducible Configuration

denoted by $ch^{'}(x)$ to each $x \in V(G) \cup F(G)$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x \in V(G) \cup F(G)} ch^{'}(x) = \sum_{x \in V(G) \cup F(G)} ch(x) = -12. \tag{2}$$

In the following, we will show that $ch^{'}(x) \geq 0$ for each $x \in V(G) \cup F(G)$, a contradiction to (2), completing the proof.

The discharging rules are defined as follows.

R1-1. Each 2-vertex receives 2 from its general 2-master or receives 1 from each of its special 2-master.

R1-2. Let f be a 3-face. If $\delta(f) \leq 3$, then f receives $\frac{3}{2}$ from each of its incident 7^{+}-vertices. Otherwise, f receives 1 from each of its incident vertices.

R1-3. Let f be a 4-face. If f is incident with two 4^{-}-vertices, then f receives 1 from each of its incident 6^{+}-vertices. Otherwise, f receives $\frac{2}{3}$ from each of its incident 5^{+}-vertices.

R1-4. For a 5-face f and its incident vertex v, f receives $\frac{1}{3}$ if $d(v) \geq 6$, $\frac{1}{5}$ if $d(v) = 5$.

let f be a face of G. Clearly, $ch'(f) = ch(f) = d(f) - 6 \geq 0$ if $d(f) \geq 6$. If $d(f) = 3$, then $ch'(f) \geq ch(f) + \min\{\frac{3}{2} \times 2, 1 \times 3\} = 0$ by R1-2 and Lemma 1. If $d(f) = 4$, then $ch'(f) \geq ch(f) + \min\{2 \times 1, \frac{2}{3} \times 3\} = 0$ by R1-3. If $d(f) = 5$, then $ch'(f) \geq ch(f) + \min\{\frac{1}{3} \times 3, \frac{1}{5} \times 5\} = 0$ by R1-4.

Let v be a vertex of G. If $d(v) = 2$, then $ch'(v) = ch(v) + 2 = 0$ by R1-1. If $d(v) = 3$, then $ch'(v) = ch(v) = 0$. If $d(v) = 4$, then v is incident with at most two 3-faces. And it follows that $ch'(v) \geq ch(v) - 2 \times 1 = 0$. If $d(v) = 5$, then $ch'(v) \geq ch(v) - \max\{3 + \frac{1}{5} \times 2, 2 + \frac{2}{3} \times 2 + \frac{1}{5}, 1 + \frac{2}{3} \times 2 + \frac{1}{5} \times 2\} = \frac{7}{15} > 0$. If $d(v) = 6$, then $ch'(v) \geq ch(v) - 4 \times 1 - 2 \times 1 = 0$. If $d(v) = 7$, then f is incident with at most four 3-faces. And if $f_3(v) = 4$, then $f_4(v) \leq 1$. Thus $ch'(v) \geq ch(v) - \max\{\frac{3}{2} \times 4 + 1 + \frac{1}{3} \times 2, \frac{3}{2} \times 3 + 3 + \frac{1}{3}\} = \frac{1}{6} > 0$.

If $d(v) = 8$, then $ch(v) = 2 \times 8 - 6 = 10$ and v is incident with at most five 3-faces by Lemma 4. If v is adjacent to no 2-vertex, then $ch'(v) \geq 10 - \max\{\frac{3}{2} \times 5 + \frac{1}{3} \times 3, \frac{3}{2} \times 4 + 4 \times 1\} = 0$.

Otherwise, v is adjacent to at least one 2-vertex. we consider the following 4-cases.

Case 1. $f_3(v) = 5$.

In this case, v must be a special 2-master by Lemma 2, then $ch'(v) \geq ch(v) - 1 - \frac{3}{2} \times 5 - \frac{1}{3} \times 3 = \frac{1}{2} > 0$.

Case 2. $f_3(v) = 4$. Various situations are illustrated in Figure 3.

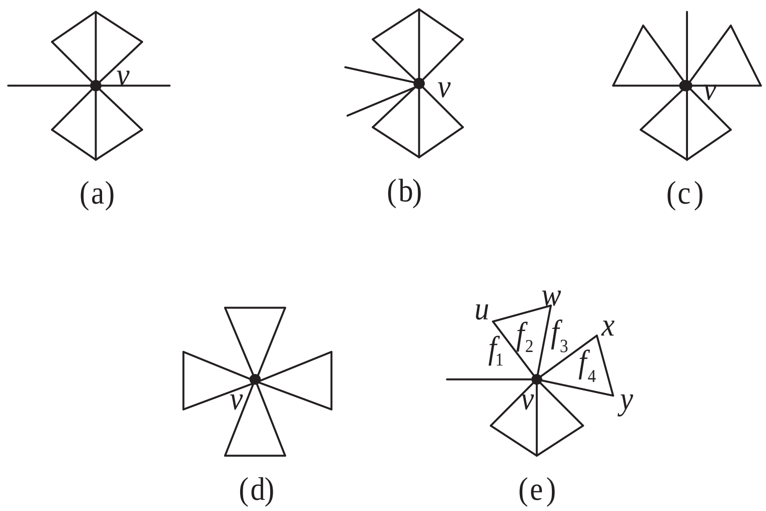
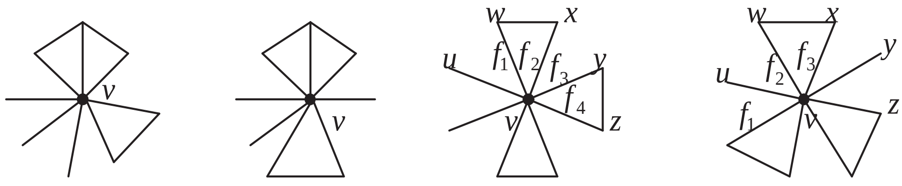


Figure 3

In Figure 3(a)-(c), $f_4(v) \leq 1$, then $ch'(v) \geq ch(v) - 2 - \frac{3}{2} \times 4 - \max\{\frac{1}{3} \times 4, \frac{1}{3} \times 3 + 1\} = 0$. In Figure 3(d), v must be a special 2-master by Lemma 2, then $ch'(v) \geq ch(v) - 1 - \frac{3}{2} - 3 \times 1 - 4 = \frac{1}{2} > 0$. In figure 3(e), $f_4(v) \leq 2$.If v is a special 2-master, then $ch'(v) \geq ch(v) - 1 - \frac{3}{2} \times 4 - 2 - \frac{1}{3} \times 2 = \frac{1}{3} > 0$. Otherwise, if f_1 is a 5^+-face, then $ch'(v) \geq ch(v) - 2 - \frac{3}{2} \times 4 - 1 - \frac{1}{3} \times 3 = 0$. If f_1 is a 4-face, then $d(u) \geq 4$ by Lemma 3. And if $\delta(f_2) \geq 4$, then $ch'(v) \geq ch(v) - 2 - \frac{3}{2} \times 3 - 1 - 1 - \frac{2}{3} - \frac{1}{3} \times 2 = \frac{1}{6} > 0$. Otherwise $d(w) = 3$, $d(u) \geq 7$. In this case, if $d(x) = 3$, then f_3 must be a 5^+-face by Lemma 2, thus $ch'(v) \geq ch(v) - 2 - \frac{3}{2} \times 4 - \frac{2}{3} - \frac{1}{3} \times 3 = \frac{1}{3} > 0$. Otherwise, $ch'(v) \geq ch(v) - 2 - \max\{\frac{3}{2} \times 4 + \frac{2}{3} \times 2 + \frac{1}{3} \times 2, \frac{3}{2} \times 3 + 1 + \frac{2}{3} + 1 + \frac{1}{3} \times 2\} = 0$.

Case 3. $f_3(v) = 3$. If v is a special 2-master, then $ch'(v) \geq ch(v) - 1 - \frac{3}{2} \times 3 - 4 - \frac{1}{3} = \frac{1}{6} > 0$. We only need to consider the case that v is a general 2-master. Various situations are illustrated in Figure 4.

In figure 4(a) and 4(b), $ch'(v) \geq ch(v) - 2 - \frac{3}{2} \times 3 - 2 - \frac{1}{3} \times 3 = \frac{1}{2} > 0$. In Figure 4(c), without loss of generality, assume $d(u) = 2$. If f_1 is a 5^+-face, then $ch'(v) \geq ch(v) -$



(a) (b) (c) (d)

Figure 4

$2-\max\{\frac{3}{2}\times 3+\frac{2}{3}+2+\frac{1}{3}\times 2, 1+\frac{3}{2}\times 2+3+\frac{1}{3}\times 2\}=\frac{1}{6}>0$. Otherwise, if f_1 is a 4-face, then $d(w)\geq 4$ by Lemma 2 and Lemma 3. In this case, if $\delta(f_2)\geq 4$, then $ch'(v)\geq ch(v)-2-\frac{3}{2}\times 2-1-\frac{2}{3}-3-\frac{1}{3}=0$. Otherwise, $d(x)=3$, $d(w)\geq 7$. If $\delta(f_4)\geq 4$, then $ch'(v)\geq ch(v)-2-\frac{3}{2}\times 2-1-\frac{2}{3}\times 2-2-\frac{1}{3}=\frac{1}{3}>0$. If $\delta(f_4)\leq 3$, then $ch'(v)\geq ch(v)-2-\max\{\frac{3}{2}\times 3+\frac{2}{3}\times 3+1+\frac{1}{3}, \frac{3}{2}\times 3+\frac{2}{3}\times 2+1+\frac{1}{3}\times 2\}=\frac{1}{6}>0$. In Figure 4 (d), without loss of generality, assume $d(u)=2$. If $d(f_1)\geq 5$, $d(f_2)\geq 5$, then $ch'(v)\geq 10-2-\frac{3}{2}\times 3-2-\frac{1}{3}\times 3=\frac{1}{2}>0$. Otherwise, assume f_2 is a 4-face, then $d(w)\geq 4$ by Lemma 2 and Lemma 3. If $\delta(f_3)\geq 4$, then $ch'(v)\geq ch(v)-2-\frac{3}{2}\times 2-1-3-\frac{2}{3}=\frac{1}{3}>0$. Otherwise, $d(x)=3$, $d(w)\geq 7$, then $ch'(v)\geq ch(v)-2-\frac{3}{2}\times 3-\frac{2}{3}-2-\frac{2}{3}=\frac{1}{6}>0$.

Case 4. $f_3(v)\leq 2$.

In this case, $ch'(v)\geq ch(v)-2-\frac{3}{2}\times 2-4-\frac{1}{3}\times 2=\frac{1}{3}>0$.

In any case, we have $ch'(x)\geq 0$ for each element $x\in V(G)\cup F(G)$, a contradiction. Hence we complete the proof of Theorem 1.

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