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Decycling Number of Circular Graphs*

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Abstract The lower and upper bounds on the decycling number of circular graph $C(n,k)$ where $k \leq \lfloor \frac{n}{2} \rfloor$ of order n are obtained. The explicit expressions of that of some classes of graphs are presented.

Keywords circular graph; decycling number; independent set

1 Introduction: the decycling number of graphs

It is well known that the cycle rank of a graph is the minimum number of edges whose removal eliminates all cycles in the graph. The parameter has a simple expression. That is, if G is a graph with p vertices, q edges and k components, then the cycle rank $\beta(G) = q - p + k$. It is an important invariant to characterize a graph. The corresponding problem of removing vertices does not have such a simple solution. It is quite difficult even for some elementary graph.

Let $G(V,E)$ be a graph. If $S \subseteq V(G)$ and $G - S$ is acyclic, then S is said to be a *decycling set* of G. The minimum order of decycling set is called the *decycling number* of G and is denoted by $\bigtriangledown(G)$. A decycling set of this order is called a $\bigtriangledown$-set. It was shown[4] that determining the decycling number of an arbitrary graph is NP-complete. The results on the decycling number of several classes of simply defined graphs can be seen in [1-2,7].

For the basic terminologies and notations, we refer the reader to [3].

2 The decycling number of circular graph

Let n and l be two positive integers with $n \geq 2l$. For any two numbers i, j where $1 \leq i,j \leq n$, define function

$$\chi(i-j) = \begin{cases} |i-j|, & \text{if } |i-j| \leq \frac{n}{2} \\ n - |i-j|, & \text{otherwise.} \end{cases}$$

A circular graph $G = C(n,l)$ of order n is one spanned by a n-circuit $C_n = (1,2,\ldots,n)$ together with the chords $(i,j) \in E(G)$ iff $\chi(j-i) = l$ ($l > 1$). Circular graphs are very

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useful for their own sake. In literature [5,6], the crossing number and some topological results of circular graphs were shown. In this paper the decycling number of circular graph is discussed.

Theorem 1 $\lceil \frac{n+1}{3} \rceil \le \bigtriangledown(C(n,l)) \le \lceil \frac{n}{2} \rceil$ *where* $n > 2l$ *and* $l > 1$.

Proof. It is easy to see that graph $C(n,l)$ is 4-regular. Firstly we prove that $\bigtriangledown(C(n,l)) \ge \lceil \frac{n+1}{3} \rceil$. Suppose S is a decycling set of graph $C(n,l)$ with order m. To make graph $C(n,l)-S$ acyclic, at most $n-m-1$ edges are allowed. That is, at least $2n-(n-m-1) = n+m+1$ edges should be removed. Let $E(S) = \{e = (u,v) \mid u \text{ or } v \in S\}$. Then $|E(S)| \le 4m$ and $|E(S)| = 4m$ if and only if S is an independent set. Then $4m \ge n+m+1$ is obtained. It follows that $m \ge \lceil \frac{n+1}{3} \rceil$.

Then we prove that $\bigtriangledown(C(n,l)) \le \lceil \frac{n}{2} \rceil$.

Case 1 n is even and l is odd. Let $S = \{x \mid x \text{ is even},\ x \le n\}$. $\forall y \in V(C(n,l)) - S$, if $(y,z) \in E(C(n,l))$, z is even. So $z \in S$. That is to say that $C(n,l) - S$ is a set of $\frac{n}{2}$ isolated vertices. So $\bigtriangledown(C(n,l)) \le \frac{n}{2}$.

Case 2 n is even and l is even. If $n = 4k$ let

$$S = \{x \mid x \text{ is even and } x \le \frac{n}{2}\} \cup \{y \mid y \text{ is odd and } \frac{n}{2} < x < n\}.$$

If $n = 4k+2$ let

$$S = \{x \mid x \text{ is even and } x < \frac{n}{2}\} \cup \{y \mid y \text{ is odd and } \frac{n}{2} < x < n\} \cup \{\frac{n}{2}+1\}.$$

All of the edges $(i, i+1)$, $i = 1, 2, \cdots, n-1$, are not in $G-S$. Now we prove that S is a decycling set of G. Assume C is a cycle in $G-S$ and C does not contain the edge $(1,n)$, then it is composed of the edges $(i, i+l)$. Without loss of generality, we can suppose that $C = \{i, i+l, i+2l, \cdots, i+xl\}$ where the numbers are taken modulo n and $i < \frac{n}{2}$. We can see that $i+xl+l-n = i$. Then i is odd, $i+xl$ is odd too but $i+xl$ belongs to the set S. So such a circuit does not exist.

Assume C is a cycle in $G-S$ and C contains the edge $(1,n)$, since $1+n-l \in S$, $n+l \equiv l \in S$, the circuit C is $\{1+xl, \cdots, 1+2l, 1+l, 1, n, n-l, n-2l, \cdots, n-yl\}$ and $1+xl+l \equiv n-yl$ which is impossible since $1+(x+1)l$ is odd and $n-yl$ is even. Then graph $C(n,l)-S$ is acyclic. So $\bigtriangledown(C(n,l)) \le \lceil \frac{n}{2} \rceil$.

Case 3 n is odd and l is odd. Let $S = \{x \mid x \text{ is even},\ x \le n\} \cup \{n\}$. In graph $C(n,l)-S$, given an odd number y ($l \le y < n-l$), it is an isolated vertex. For an odd number y ($0 < y < l$), it is adjacent to only one vertex $n+y-l$. That is, it is an articulate vertex. For the same reason, the vertex y ($n-l < y < n$) is adjacent to only one vertex $y+l-n$. Graph $C(n,l)-S$ is acyclic. So $\bigtriangledown(C(n,l)) \le \lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil$.

Case 4 n is odd and l is even. Let $S = \{x \mid x \text{ is even},\ x \le n\} \cup \{n\}$. For any $y \in V(C(n,l)) - S$, at least two edges of E_y are removed. If there is a circuit C in graph $(C(n,l))-S$, C must be composed of such edges as $(i, i+l)$ where the numbers are modulo n. Such an edge (x,y) ($n-l \le x \le n$, $0 \le y \le l$) is inevitable which contradicts the fact that x and y should be odd at the same time. So $C(n,l)-S$ is acyclic. Then $\bigtriangledown(C(n,l)) \le \lceil \frac{n}{2} \rceil$.

□

From the proof of this theorem the following corollary is direct:

Corollary 2 *Suppose $n \equiv 2$ $(mod\ 3)$ and $\bigtriangledown(C(n,l)) = |S| = \frac{n+1}{3}$ for certain l where S is a decycling set and $n \neq 2l$. Then the decycling set is an independent set and the graph $G-S$ is a tree(connected and acyclic).*

Lemma 3[2] *If G and H are homomorphic graphs, then $\bigtriangledown(G) = \bigtriangledown(H)$.*

In the following paper, we determine the explicit expression of $\bigtriangledown$-set for certain circular graphs. At first if $n = 2l$ then graph $C(n,l)$ is 3-regular. We discuss the decycling set of $C(n,l)$.

Lemma 4 $\bigtriangledown(C(n,l)) \geq \lceil \frac{l+1}{2} \rceil$ *where $n = 2l$.*

Proof. Suppose S is a decycling set with $|S| = m$ of graph $C(n,l)$. The size of graph $C(n,l) - S$ is at most $l - m - 1$. That is to say, at least $l + m + 1$ edges should be eliminated. Then we get $3m \geq l + m + 1$ which follows that $m \geq \lceil \frac{l+1}{2} \rceil$. □

Theorem 5 $\bigtriangledown(C(n,l)) = \lceil \frac{l+1}{2} \rceil$ *where $l \geq 2$ and $n = 2l$.*

Proof. Easily to see that graph $C(4,2)$ is homomorphic to the complete graph K_4. From [6], we know that $\bigtriangledown(C(4,2)) = 2$.

Suppose A is a set of edges of graph G. Then $\bigtriangledown(G-A) \leq \bigtriangledown(G)$. For any edge $(i,j) \in E(C(n,l))$ where $|i-j| \neq 1$ graph $C(n,n) - (i,j)$ is homomorphic to graph $C(n-2,l-1)$. From Lemma 3, $\bigtriangledown(C(n-2,l-1)) \leq \bigtriangledown(C(n,l)) \leq \bigtriangledown(C(n+2,l+1))$.

By induction on the number l. When $l = 3$, the circular graph $C(6,3)$ is isomorphic to $K_{3,3}$, so $\bigtriangledown(C(6,3)) = 2 = \lceil \frac{3+1}{2} \rceil$. If $l = 4$, from Lemma 4, $\bigtriangledown(C(8,4)) \geq 3$. Suppose $S = \{1,3,5\}$, $C(8,4)$ is acyclic. Then $\bigtriangledown(C(8,4)) = 3 = \lceil \frac{4+1}{2} \rceil$.

Suppose when $nl = k$ (k is even) this theorem holds. We can get that $\frac{k}{2} + 1 \leq \bigtriangledown(C(2k+2,k+1))$. Let $S = \{i\ |i$ is odd, and $i \leq k+1\}$. The induced graph $C(2k+2,k+1) - S$ is a tree. So $\bigtriangledown(C(2k+2,k+1)) = \frac{k}{2} + 1 = \lceil \frac{k+2}{2} \rceil$.

Suppose when $n = k$ (k is odd) this theorem holds. It is got that $\frac{k-1}{2} + 1 \leq \bigtriangledown(C(2k+2,k+1))$. On the other hand, from Lemma 4 $\bigtriangledown(C(2k+2,k+1)) \geq \frac{k-1}{2} + 2$. Let $S = \{i\ |i$ is odd, and $i \leq k+1\}$. The induced graph $C(2k+2,k+1) - S$ is a tree. So $\bigtriangledown(C(2k+2,k+1)) = \frac{k+1}{2} + 1 = \lceil \frac{k+2}{2} \rceil$.

This theorem follows. □

Most circular graphs $C(n,l)$ is 4-regular. Then we cinsider 4-regular circulat graph. If $= 2$ we have the following result:

Theorem 6

$$\bigtriangledown(C(n,2)) = \begin{cases} \lceil \frac{n+1}{3} \rceil + 1, & \text{if } n \equiv 2,5 \text{ mod } 6 \\ \lceil \frac{n+1}{3} \rceil, & \text{otherwise.} \end{cases}$$

where $n \geq 5$.

Proof. When $n = 5$, $C(5,2)$ is isomorphic to K_5. And $\bigtriangledown(C(5,2)) = 3 = \lceil \frac{5+1}{3} \rceil + 1$.

Then we prove the case when $n \geq 6$. First from Theorem 1, we know $\bigtriangledown(C(n,2)) \geq \lceil \frac{n+1}{3} \rceil$.

Case 1 $n = 6k$ where k is a positive integer. On one hand, $\bigtriangledown(C(6k,2)) \geq 2k+1$. Suppose $S = \{3i \mid i = 1,2\ldots,2k\} \cup \{1\}$. Easy to see that $C(6k,2) - S$ is a path of length $4k-2$. Then $\bigtriangledown(C(6k,2)) = 2k+1 = \lceil \frac{n+1}{3} \rceil$.

Case 2 $n = 6k+2$ where k is a positive integer. It is easy to see that $\bigtriangledown(C(6k,2)) \geq 2k+1$. We say that $\bigtriangledown(C(6k+2,2)) \geq 2k+2$. Otherwise suppose that S is a decycling set with order $2k+1$ of graph $C(6k+2,2)$. From Corollary 2, $|E(S)| = 4k+4$. Select any $2k+1$ vertices from $6k+2$ vertices. Since $\frac{6k+2}{2k+1} < 3$, the occurrence of such pairs of vertices as i and $i+1$ or i and $i+2$ is inevitable. So $|E(S)| = 4k+4$ is impossible. Then we get that $\bigtriangledown(C(6k,2)) \geq 2k+2$. On the other hand, let $S = \bigcup_{i=1}^{2k}\{3i\}\bigcup\{6k+1, 6k+2\}$. The graph $C(6k,2) - S$ is a path of length $4k-1$. Then $\bigtriangledown(C(6k+2,2)) = 2k+2 = \lceil\frac{6k+3}{3}\rceil + 1$.

Case 3 $n = 6k+4$ where k is a positive integer. On one hand, $\bigtriangledown(C(6k+4,2)) \geq 2k+2$. On the other hand let $S = \bigcup_{i=1}^{2k+1}\{3i\}\bigcup\{6k+4\}$. The graph $C(6k,2) - S$ is a path of length $4k+1$. So $\bigtriangledown(C(6k+4,2)) = 2k+2 = \lceil\frac{6k+5}{3}\rceil$.

Case 4 $n = 6k+1$ where k is a positive integer. From Theorem 1, $\bigtriangledown(C(6k+1,2)) \geq 2k+1$. Then let $S = \bigcup_{i=1}^{2k}\{3i\}\bigcup\{6k+1\}$. One can see that graph $C(6k+1,2) - S$ is a path of length $4k-1$. So $\bigtriangledown(C(6k+1,2)) = 2k+1 = \lceil\frac{6k+2}{3}\rceil$.

Case 5 $n = 6k+3$ where k is a positive integer. On one hand, Ffom Theorem 1 $\bigtriangledown(C(6k+3,2)) \geq 2k+2$. On the other hand let $S = \bigcup_{i=1}^{2k+1}\{3i\}\bigcup\{1\}$. The graph $C(6k+3,2) - S$ is a path of length $4k$. So $\bigtriangledown(C(6k+3,2)) = 2k+2 = \lceil\frac{6k+4}{3}\rceil$.

Case 6 $n = 6k+5$ where k is a positive integer. On one hand, $\bigtriangledown(C(6k+5,2)) \geq 2k+2$. In graph $C(6k+5,2)$, any $2k+2$ vertices can be incident to at most $8k+7$ edges since the occurrence of such pair of vertices as $(i,i+1)$ or $(i,i+2)$ is inescapable. So $\bigtriangledown(C(6k+5,2)) \geq 2k+3$. Let $S = \bigcup_{i=1}^{2k+1}\{3i\}\bigcup\{6k+4, 6k+5\}$. The graph $C(6k+5,2) - S$ is acyclic. Then $\bigtriangledown(C(6k+5,2)) = 2k+3 = \lceil\frac{6k+6}{3}\rceil + 1$.

This theorem is found. □

Theorem 7

$$\bigtriangledown(C(n,3)) = \begin{cases} \lceil\frac{n+1}{3}\rceil + 1, & \text{if } n = 3k+2 \text{ and } k \text{ is odd,} \\ \lceil\frac{n+1}{3}\rceil, & \text{otherwise.} \end{cases}$$

where $n \geq 7$.

Proof First from Theorem 1, it is known that $\bigtriangledown(C(n,3)) \geq \lceil\frac{n+1}{3}\rceil$.

Case 1 $n = 3k$ where k is a positive integer. Suppose that $S = \{a_t, n-1\}$ where $a_1 = 1$, $a_t < n$ and

$$a_{t+1} = \begin{cases} a_t + 2, & \text{if } t \text{ is odd,} \\ a_t + 4, & \text{if } t \text{ is even.} \end{cases}$$

There is no cycle in graph $C(3k,3) - S$, so $\bigtriangledown(C(n,3)) = \lceil\frac{n+1}{3}\rceil$ when $n = 3k$.

Case 2 $n = 3k+1$ where k is a positive integer. Suppose that $S = \{a_t, n-1\}$ where $a_1 = 1$, $a_t < n$ and

$$a_{t+1} = \begin{cases} a_t + 2, & \text{if } t \text{ is odd,} \\ a_t + 4, & \text{if } t \text{ is even.} \end{cases}$$

The graph $C(3k+1,3) - \{a_t\}$ when k is odd is acyclic, so is graph $C(3k+1,3) - S$ when k is even. Summarizing above, it is obtained that $\bigtriangledown(C(n,3)) = \lceil\frac{n+1}{3}\rceil$ when $n = 3k+1$.

Case 3 $n = 3k+2$ where k is a positive even integer. Suppose that $S = \{a_t\}$ where $a_1 = 1$, $a_t < n$ and

$$a_{t+1} = \begin{cases} a_t + 2, & \text{if } t \text{ is odd,} \\ a_t + 4, & \text{if } t \text{ is even.} \end{cases}$$

Graph $C(3k+2) - S$ is acyclic. So $\bigtriangledown(C(n,3)) = \lceil \frac{n+1}{3} \rceil$ when $n = 3k+2$.

Case 4 $n = 3k+2$ where k is a positive odd integer. It can be seen that $n = 6 \times \frac{k-1}{2} + 5 = 6m+5$. We prove that $\bigtriangledown(C(n,3)) \neq \lceil \frac{n+1}{3} \rceil = 2m+2$. By contradiction. Suppose S is a decycling set of $C(n,3)$ and $|S| = 2m+2$. Make the numbers in S with ascending order, that is, $S = \{b_1, b_2, \cdots, b_{|S|}\}$ where $b_1 < b_2 < \cdots < b_{|S|}$. Since S is a decycling set, then $|E(S)| = 8m+8$. We know S is an independent set. For any two vertices b_i and b_{i+1} in S, $\chi(b_{i+1} - b_i) \neq 1$ or 3 and $\chi(b_{i+1} - b_i) \leq 4$. Suppose there are x pairs of numbers $(b_i, b_{i+1}) \in S$ such that $\chi(b_{b+1} - b_i) = 2$ and y pairs of numbers $(b_i, b_{i+1}) \in S$ such that $\chi(b_{i+1} - b_i) = 4$. Then two equation are derived: $x + y = 2m+2$ (the number of the vertices whose removal is necessary) and $x + 3y = 4m+3$. It follows that $2y = 2m+1$. This is impossible. So $\bigtriangledown(C(n,3)) \neq \lceil \frac{n+1}{3} \rceil = 2m+2$. That is $\bigtriangledown(C(6m+5,3)) \geq 2m+3$. Let $S = \{a_t, n-1\}$ where $a_1 = 1$, $a_t < n$ and

$$a_{t+1} = \begin{cases} a_t + 2, & \text{if } t \text{ is odd,} \\ a_t + 4, & \text{if } t \text{ is even.} \end{cases}$$

The graph $C(3k,2) - S$ is acyclic when k is a positive odd integer, so $\bigtriangledown(C(3k+2,3)) = \lceil \frac{3k+3}{3} \rceil + 1 = k+2$.

This theorem is found. □

To obtain more results on decycling number, we introduce two operators. They are used in the following proof.

Given a labeled circular graph G with order n, suppose S is a set of vertices $S = \{i_1, i_2, \cdots, i_l\}$. Define two operators δ and δ' such that $\delta S = \{i_j - i_{j-1} - 1, i_1 - i_l - 1 \mid j = 2, 3, \cdots, l\}$ and $\delta' S = \{i_j - i_{j-1} - 1 \mid j = 2, 3, \cdots, l\}$ where the elements read modulo n. From sets $\delta S = \{p_1, p_2, \cdots, p_l\}$ or $\delta' S = \{p_1, p_2, \cdots p_{l-1}\}$ it is able to get the set S too. But the set S is different according to the choice of number i_1.

For example, suppose G is the circular graph $C(17,8)$ and $S = \{3,6,7,9,10,15,17\}$. Then $\delta S = \{2,0,1,0,4,1,2\}$ and $\delta' S = \{2,0,1,0,4,1\}$. If $\delta S = \{2,1,1,0,3,1,2\}$ and suppose $i_1 = 1$ the set S could be $\{1,4,6,8,9,13,15\}$.

Theorem 8

$$\bigtriangledown(C(n,4)) = \begin{cases} \lceil \frac{n+1}{3} \rceil + 1, & n = 3k+2 \text{ and } k \text{ is a positive integer,} \\ \lceil \frac{n+1}{3} \rceil, & \text{otherwise.} \end{cases}$$

where $n \geq 9$.

Proof From Theorem 1, we know that $\bigtriangledown(C(n,4)) \geq \lceil \frac{n+1}{3} \rceil$. To get the lower bound, three cases are divided.

Case 1 $n = 3k$ where k is a positive integer. Let $S_0 = \{3i \mid i = 1, 2, \cdots, k\}$. In graph $C(3k,4) - S_0$, for the vertex $v = 3i+1$, two edges $(3i+1, 3i)$ and $(3i+1, 3i-3)$ are removed. For the vertex $v = 3i+2$, the edges $(3i+2, 3i+3)$ and $(3i+2, 3i+6)$ are removed. So the graph $C(3k,4) - S_0$ is a cycle of length $2k$. Removing any vertex of graph

$C(3k,4)-S_0$, without loss of generality, suppose the vertex 1 is removed, the resultant graph is acyclic. So $S=S_0\cup\{1\}$ is a $\bigtriangledown$-set. Then $\bigtriangledown(C(3k,4))=k+1=\lceil\frac{n+1}{3}\rceil$.

Case 2 $n=3k+1$ where k is a positive integer. Let $S=\{3i\mid i=1,2,\cdots,k\}\cup\{1\}$. In graph $C(3k,4)-S$, for the vertex $v=3i+1$ $(1\le i<k-1)$, it is incident to two edges $(3i+1,3i+2)$ and $(3i+1,3i+5)$. For the vertex $v=3i+2$ $(i\ge 2)$, it is incident to two edges $(3i+2,3i+1)$ and $(3i+2,3(i-1)+1)$. The vertices $2,n,5,n-3$ are articulated and the edges that they are incident with are $(2,n-2)$, $(n,4)$, $(5,1)$ and $(n-3,1)$ respectively. So the graph $C(3k,4)-S$ is acyclic. Then $\bigtriangledown(C(3k+1,4))=k+1=\lceil\frac{n+1}{3}\rceil$.

Case 3 $n=3k+2$ where k is a positive integer. First we say that $\bigtriangledown(C(3k+2,4))\ne k+1$. Otherwise suppose that S is a decycling set of $k+1$ vertices. From Corollary 2, $|E(S)|=4k+4$. Assume the numbers in S are sorted with ascending order. For any vertex a and its successor b in S , $\chi(a-b)=2$, $\chi(a-b)=3$ or $\chi(a-b)=5$. Suppose there are x pairs of numbers (a,b) such that $\chi(a-b)=2$, y pairs of numbers (c,d) such that $\chi(c-d)=3$ and z pairs of numbers (e,f) such that $\chi(e-f)=5$. Two equations are followed that $x+y+z=k+1$ and $x+2y+4z=2k+1$.

If $z=0$ then $y=k$ and $x=1$. Suppose $i,i+2$ are two numbers of S. Then the vertex $i+1$ in graph $C(n,4)-S$ is an isolated vertex which contradicts Corollary 2.

Then we have $z\ne 0$. Suppose the vertices $i,i+5\in S$ and the vertices $i+1,i+2,i+3,i+4\notin S$. One can see that $i+6$ and $i+9\notin S$ since set S is an independent set. The vertex $i+7\in S$ because set S is a decycling set. And it forces vertex $i+8\notin S$. The vertex $i+10\in S$ otherwise a circuit $\{i+2,i+3,i+4,i+8,i+9,i+10,i+6\}$ appears which contradicts that set S is a decycling set. Then the vertices $i+11$ and $i+14\notin S$. In order to get an acyclic graph, one of the vertices $i+12,i+13$ should be removed.

If the vertex $i+12\in S$ one can see that the vertices $i+13,i+14,i+15,i+16\notin S$. Another pair of numbers $a,b\in S$ such that $\chi(a-b)=5$ are obtained. That is to say a sequence T_1 with $\delta' T_1=\{4,1,2,1\}$ is available.

If the vertex $i+13\in S$ one can obtain a sequence T_2 with $\delta' T_2=\{4,1,2,2,2,\cdots,1\}$.

The difference of T_1 and T_2 lies in the number of pairs (a,b) with $\chi(a-b)=2$. Whatever the sequence in set S is, the order of the graph is a multiple of 3 which contradicts that $n=3k+2$.

In the proof of the case $z\ne 0$, $n\ge 12$. If $n=11$, we know $z\ne 0$. Otherwise suppose $(1,6)\in S$, a cycle $\{3,4,11,10\}$ occurs.

Summarizing above one know that $\bigtriangledown(C(3k+2,4))\ge k+2$. Suppose $S=\{3i\mid i=1,2,\cdots,k\}\cup\{1,n\}$. The graph $C(n,4)-S$ is acyclic. Then $\bigtriangledown(C(3k+2,4))=k+2$. □

Theorem 9 *Suppose* $n=3k$, $l=3m-1$ *and* $(k,m)=1$ *where* $k\ge 3m$. *Then* $\bigtriangledown(C(n,l))=k+1=\lceil\frac{n+1}{3}\rceil$.

Proof. At first let $S_0=\{3i\mid i=1,2,\cdots,k\}$. For any vertex $3i+1$, $3i+1+l=3(i+m)\in S_0$, then $d_{G-S_0}(3i+1)\le 2$. And $3i+1-l=3(i-m)+2\notin S_0$, then $d_{G-S_0}(3i+1)=2$. For the vertex $3i+2$, $d_{G-S_0}(3i+2)=2$ since $3i+2+l=3(i+m)+1\notin S_0$ and $3i+2-l=3(i-m+1)\in S_0$ then $d_{G-S_0}(3i+2)=2$.

Now we prove that $G-S_0$ is a circuit but not the disjoint union of some circuits. It suffices to prove that the set $A=B=\{1,4,7,10\cdots,n-2\}$ where $A=\{1,3m+1,6m+1,9m+1\cdots\}$ and the elements are modulo n.

In set A, suppose there are two elements $k_1\times 3m+1\equiv k_2\times 3m+1$. Without loss of generality we can assume that $k_1>k_2$. There exists a positive number x such that

$k_1 \times 3m+1 = k_2 \times 3m+1+x \times 3k$ which follows that $(k_1-k_2)m = xk$. The left can be divided by m, so xk is a multiple of m. On the other hand, $1 \leq |k_1-k_2| \leq k$ which follows that $\frac{xk}{m} \leq k$. Then we can get $x \leq m$ which contradicts that xk is a multiple of m. Then every two elements in A are different.

Let $S = S_0 \cup \{1\}$. The graph $G-S$ is a path of length $2k-1$. □

Theorem 10 *Suppose* $n = 3k$, $l = 3m-1$ *and* $(k,m) = 2$ *where* $k \geq 3m$. *Then* $\bigtriangledown(C(n,l)) = k+1 = \lceil \frac{n+1}{3} \rceil$.

Proof. At first let $S_0 = \{3i \mid i = 1,2,\cdots,k\}$. For any vertex $3i+1$, $3i+1+l = 3(i+m) \in S_0$, then $d_{G-S_0}(3i+1) \leq 2$. And $3i+1-l = 3(i-m)+2 \notin S_0$, then $d_{G-S_0}(3i+1) = 2$. For the vertex $3i+2$, $d_{G-S_0}(3i+2) = 2$ since $3i+2+l = 3(i+m)+1 \notin S_0$ and $3i+2-l = 3(i-m+1) \in S_0$ then $d_{G-S_0}(3i+2) = 2$. Removing the vertices in set S_0, two circuits are obtained $C_1 = \{1,2,2+l,3+l,\cdots,n-l,n-l+1\}$ and $C_2 = \{n-1,n-2,\cdots,l-1\}$. Let $S = S_0 - \{n\} + \{1,n-1\}$. It can be verified that graph $C(n,l)-S$ is acyclic. So $\bigtriangledown(C(n,l)) = k+1$. □

From these theorems we know the lower bound in Theorem 1 is best possible since it is reached.

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