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Global Synchronization for General Delayed Dynamical Networks

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Abstract In this paper, the problem of global synchronization for a class of general delayed dynamical networks with non-symmetric coupling is dealt with. The approach taken in the paper is to estimate the error state variable between any two models of the networks directly. By using a Lyapunov-Krasovskii function and comparison theorem, some simple and efficient criteria for the stability of synchronization manifold are derived. It should be pointed out that no matrix diagonalization technique or Kronecker product is involved through derivation of all the synchronization criteria.

Keywords Synchronization; Delayed dynamical networks; Lyapunov-Krasovskii function; Stability theorem

1 Introduction

Recently synchronization of the delayed dynamical networks with linearly coupled has gain a lot of attentions of the researchers. The main reason is that the time delay in signal transition is a vary familiar phenomenon ,also the effect of time delays can induce complex dynamics of some dynamical systems, such as stability, existence of periodic, oscillatory ,chaos behaviors and so on. Moreover those time delays often influence the synchronization motion and the stability of synchronization motion. Also, synchronization of delayed dynamical complex networks has been extensively studied due to its theoretical importance and practical applications[1-7]. Li and Chen[1] presented a uniform delayed model and derived some synchronization criteria for both delay- independent and delay-dependent exponential stability of synchronization state. By using Kronecker product and stability theorem, there are many existing studies related to large-scale complex delayed dynamical networks with symmetric coupling and global synchronization criteria are presented [8-11,17]. In addition, such complex network model with time-varying delay or distributed delays even with impulsive effects are also discussed [12-16]. More recently some authors are concerned with synchronization problems of complex networks with non-symmetric coupling, some synchronization criteria are also derived[17-18].

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In this paper, the problem of global synchronization for a class of general delayed dynamical networks with non-symmetric coupling is dealt with. The approach taken in the paper is to estimate the error state variable between any two models of the networks directly. By using a Lyapunov-Krasovskii function and comparison theorem, some simple and efficient criteria for the stability of synchronization manifold are derived. It should be pointed out that no matrix diagonalization technique or Kronecker product is involved through derivation of all the synchronization criteria.

2 Preliminaries

Considering a delayed dynamical network with N diffusively coupled identical nodes, we described the stated equations of the whole network as

$$\dot{x}_i(t) = Ax_i(t) + \sum_{k=1}^{N} g_{ik}\Gamma x_k(t-\tau) + f(t, x_i(t)),\ i = 1, 2, \cdots, N, \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^N \in R^n$ is the state variable of the ith node.$\Gamma \in M_n(R)$ is the inner-coupling matrix which describe the individual coupling between two connected nodes of network , $\tau \geq 0$ is the coupling time delay. $g_{ik} \in R$ describe the coupling strength from node k to node $i(k \neq i), g_{ik} \geq 0$, $f : R \times R^n \times R^n \to R^n$ is continuously vector-valued function with respect to the second variable and third variable, and $A \in M_n(R)$ describe the dynamics of an individual node .

Throughout, we assumed that $g_{ii} = -\sum_{k=1,k\neq i}^{N} g_{ik}, i = 1, 2, \cdots, N$ which imply that the row sum of matrix $G = (g_{ik}) \in M_N(R)$are all zero. We always assume that the nonlinear vector-valued function f satisfy uniformly global Lipschitz condition i.e.,

(H) For any $x = (x_1, x_2, \cdots, x_n)^N \in R^n, y = (y_1, y_2, \cdots, y_n)^N \in R^n$ and positive definite matrix P, there exists positive definite matrix D, such that

$$(x-y)^T P[f(t,x) - f(t,y)] \leq (x-y)^T D(x-y). \tag{2}$$

Applied the existence and uniqueness theorem [2] in functional differential equations theory, the Eq.(1) has a unique solution with respect to initial condition given by $x_i(t) = \varphi_i(t) \in C([-\tau, 0], R^n)$, where $C = C([-\tau, 0], R^n)$is a Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^n with the norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$.

Now some definitions of with respect to global synchronization are introduced as follows.

Definition 1. The hyperplane

$$\{(x_1^T(t), x_2^T(t), \cdots, x_n^T(t))^T : x_i(t) = x_j(t), i, j, \cdots, N\}, \tag{3}$$

is said to the synchronization state of the delayed dynamical network (1).

Obviously, the synchronization state $S(t) = (s^T(t), s^T(t), \cdots, s^T(t))^T$ of delayed dynamical network (1) satisfies

$$\dot{s}(t) = As(t) + f(t, s(t)). \tag{4}$$

Definition 2. Synchronization state $S(t)$ of delayed dynamical network (1) is said to be globally exponential stable, if there exist constants $\lambda > 0$ and $K > 0$, for all $\varphi_i(t) \in C([-\tau,0],R^n)$ such that

$$\| x_i(t) - x_j(t) \| \leq Ke^{-\lambda(t-t_0)}, i,j = 1,2,\cdots,N, \tag{5}$$

holds for all $t \geq t_0$,where $\| \cdot \|$ is the Euclidean norm.

Lemma 1. For any two n-dimensional vectors x and y, and a positive definite matrix $Q \in M_n(R)$,

$$x^T y + y^T x \leq x^T Q^{-1} x + y^T Qy. \tag{6}$$

Lemma 2. (Zhou and Chen [11]). Let $v(t) > 0$ for $t \in R, \tau \in [0,+\infty)$ and $t_0 \in R$. Suppose that

$$\dot{v}(t) \leq -av(t) + b \sup_{t-\tau \leq s \leq t} v(s), \tag{7}$$

for $t > t_0$. If $a > b > 0$, then there exist constants $K > 0$ and $\lambda > 0$, such that

$$v(t) \leq Ke^{-\lambda(t-t_0)}, fort > t_0. \tag{8}$$

3 Synchronization in delayed dynamical networks

Theorem 1. Assume that (H) holds, and there exist two $n \times n$ positive definite matrix $P,Q \succ 0$, such that

$$PA + A^T P + D - (g_{ii} + g_{jj}) P\Gamma Q \Gamma^T P + (N-1)(g_{ij} + g_{ji}) Q^{-1} \preceq -\varepsilon I_n, \tag{9}$$

Then the synchronization state $S(t)$ of the delayed networks (1) is exponentially stable.

Proof. Define Lyapunov-Krasovskii functions as follows

$$\begin{aligned} V_{ij}(t) &= (x_i(t) - x_j(t))^T P(x_i(t) - x_j(t)) \\ &\quad + (N-1)(g_{ij} + g_{ji}) \int_{t-\tau}^{t} (x_i(s) - x_j(s))^T Q^{-1} (x_i(s) - x_j(s)) ds,\ i,j = 1,2,\cdots,N, \end{aligned}$$

Denote $V(t) = \sum_{i,j=1}^{N} V_{ij}(t)$.

The derivative of $V_{ij}(t)$ along system (1) is

$$\begin{aligned} \dot{V}_{ij}(t) &= (x_i(t) - x_j(t))^T (PA + A^T P)(x_i(t) - x_j(t)) + (x_i(t) - x_j(t))^T P[f(t,x_i(t)) - f(t,x_j(t))] \\ &+ [f(t,x_i(t)) - f(t,x_j(t))]^T P(x_i(t) - x_j(t)) + (x_i(t) - x_j(t))^T P \sum_{k=1}^{N} (g_{ik} - g_{jk}) \Gamma x_k(t-\tau) \\ &+ \sum_{k=1}^{N} (g_{ik} - g_{jk}) x_k^T(t-\tau) \Gamma^T P(x_i(t) - x_j(t)) + (N-1)(g_{ij} + g_{ji})(x_i(t) - x_j(t))^T Q^{-1} \\ &\quad (x_i(t) - x_j(t)) - (N-1)(g_{ij} + g_{ji})(x_i(t-\tau) - x_j(t-\tau))^T Q^{-1} (x_i(t-\tau) - x_j(t-\tau)). \end{aligned}$$

$\dot{V}(t) = \sum_{i,j=1}^{N} \dot{V}_{ij}(t)$.

Let

$$\sigma_1(t) = \sum_{i,j=1}^{N} (x_i(t) - x_j(t))^T P \sum_{k=1}^{N} (g_{ik} - g_{jk}) \Gamma x_k(t-\tau), \tag{10}$$

$$\sigma_2(t) = \sum_{i,j=1}^{N} \sum_{k=1}^{N} (g_{ik} - g_{jk}) x_k^T(t-\tau) \Gamma^T P (x_i(t) - x_j(t)). \tag{11}$$

Then

$$\begin{aligned} \sigma_1(t) &= 2\sum_{i,k=1}^{N} [N g_{ik} - (\sum_{j=1}^{N} g_{jk})] x_i^T(t) P \Gamma x_k(t-\tau) \\ &= 2\sum_{i,j,k=1}^{N} g_{jk} (x_j(t) - x_i(t))^T P \Gamma (x_k(t-\tau) - x_j(t-\tau)). \end{aligned}$$

Similarly,

$$\sigma_2(t) = 2\sum_{i,j,k=1}^{N} g_{jk} (x_k(t-\tau) - x_j(t-\tau))^T \Gamma^T P (x_j(t) - x_i(t)).$$

Applied lemma 1 and $\sum_{k=1}^{N} g_{jk} = 0, j = 1, 2, \cdots, N$, Then

$$\begin{aligned} \sigma_1(t) + \sigma_2(t) &\leq 2\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1, j\neq i, j\neq k}^{N} g_{jk} [(x_i(t) - x_j(t))^T P \Gamma Q \Gamma^T P (x_i(t) - x_j(t)) \\ &\quad + (x_k(t-\tau) - x_j(t-\tau))^T Q^{-1} (x_k(t-\tau) - x_j(t-\tau))] \\ &= 2\sum_{i,j=1}^{N} (\sum_{k=1, k\neq j}^{N} g_{jk}) (x_i(t) - x_j(t))^T P \Gamma Q \Gamma^T P (x_i(t) - x_j(t)) \\ &+ 2\sum_{k,j=1}^{N} (\sum_{i=1, i\neq j}^{N} g_{jk}) (x_k(t-\tau) - x_j(t-\tau))^T Q^{-1} (x_k(t-\tau) - x_j(t-\tau)) \\ &= -\sum_{i,j=1}^{N} (g_{ii} + g_{jj}) (x_i(t) - x_j(t))^T P \Gamma Q \Gamma^T P (x_i(t) - x_j(t)) \\ &+ (N-1)\sum_{k,j=1}^{N} (g_{jk} + g_{kj}) (x_k(t-\tau) - x_j(t-\tau))^T Q^{-1} (x_k(t-\tau) - x_j(t-\tau)). \end{aligned}$$

From the condition (H),

$$\begin{aligned} &\sum_{i,j=1}^{N} \{(x_i(t) - x_j(t))^T P [f(t, x_i(t)) - f(t, x_j(t))] + [f(t, x_i(t)) - f(t, x_j(t))]^T P (x_i(t) - x_j(t))\} \\ \leq &\sum_{i,j=1}^{N} (x_i(t) - x_j(t))^T D (x_i(t) - x_j(t)). \end{aligned}$$

From (4),(5),(10),(11), it is easy to obtain,

$$\begin{aligned}\dot{V}(t) &\leq \sum_{i,j=1}^{N}(x_i(t)-x_j(t))^T(PA+A^TP+D-(g_{ii}+g_{jj})P\Gamma Q\Gamma^TP \\ &\quad +(N-1)(g_{ij}+g_{ji})Q^{-1})(x_i(t)-x_j(t)) \\ &\leq -\varepsilon\sum_{i,j=1}^{N}(x_i(t)-x_j(t))^T(x_i(t)-x_j(t)).\end{aligned}$$

It then follows that the synchronization state $S(t)$ of the delayed network (1) is exponentially stable.

Theorem 2. Assume that (H) holds, and there exist $n\times n$ positive definite matrix $P\succ 0$ and a positive constant β, such that

$$PA+A^TP+D-(g_{ii}+g_{jj})P\Gamma P^{-1}\Gamma^TP \preceq -\beta P, i,j=1,2,\cdots,N, \tag{12}$$

and

$$\tau < \frac{\beta-2(N-1)\bar{g}}{(N-1)\bar{g}(1+2[\beta^2+2(N-1)^2\bar{g}^2])}, \tag{13}$$

Then the synchronization state $S(t)$ of the delayed networks (1) is exponentially stable.

Proof. Define

$$W_1 = \sum_{i,j=1}^{N}(x_i(t)-x_j(t))^TP(x_i(t)-x_j(t)), P=R^TR \tag{14}$$

By using the proof of theorem 1,

$$\begin{aligned}\dot{W}_1(t) &\leq \sum_{i,j=1}^{N}(x_i(t)-x_j(t))^T(PA+A^TP+D-(g_{ii}+g_{jj})P\Gamma P^{-1}\Gamma^TP)(x_i(t)-x_j(t)) \\ &\quad +(N-1)\sum_{i,j=1}^{N}(g_{ji}+g_{ij})(x_i(t-\tau)-x_j(t-\tau))^TP(x_i(t-\tau)-x_j(t-\tau)) \\ &\leq -\beta W_1(t)+(N-1)\bar{g}W_1(t-\tau),\end{aligned}$$

where $2\bar{g} = \max_{1\leq i,j\leq N, i\neq j}(g_{ji}+g_{ij}) > 0$.

Let $W = W(t)(t\geq t_0)$ be the solution of following initial value problem

$$\dot{W}(t) = -\beta W(t)+2(N-1)\bar{g}W(t-\tau), W(t)=W_1(t), t\in[t_0-\tau,t_0] \tag{15}$$

By using comparison theorem of differential equations,$W_1(t)\leq W(t)$, for $t\geq t_0$.

Rewrite equation (15) as

$$\dot{W}(t) = -(\beta-2(N-1)\bar{g})W(t)-2(N-1)\bar{g}\int_{t-\tau}^{t}\dot{W}(s)ds, W(t)=W_1(t), t\in[t_0-\tau,t_0] \tag{16}$$

Denote $W_2(t)=\frac{1}{2}W^2(t)$

Then

$$\begin{aligned}
\dot{W}_2(t) &= -(\beta - 2(N-1)\bar{g})W^2(t) - 2(N-1)\bar{g}W(t)\int_{t-\tau}^{t}\dot{W}(s)ds \\
&\leq -(\beta - 2(N-1)\bar{g})W^2(t) + (N-1)\bar{g}\int_{t-\tau}^{t}[W^2(t) + \dot{W}^2(s)]ds \\
&\leq -(\beta - 2(N-1)\bar{g} - (N-1)\bar{g}\tau)W^2(t) + (N-1)\bar{g}\tau \sup_{t-\tau\leq s\leq t}\dot{W}^2(s) \\
&= -(\beta - 2(N-1)\bar{g} - (N-1)\bar{g}\tau)W^2(t) \\
&\quad +(N-1)\bar{g}\tau \sup_{t-\tau\leq s\leq t}[-\beta W(s) + 2(N-1)\bar{g}W(s-\tau)]^2 \\
&\leq -(\beta - 2(N-1)\bar{g} - (N-1)\bar{g}\tau)W^2(t) + 2(N-1)\bar{g}\tau[\beta^2 + 2(N-1)^2\bar{g}^2] \sup_{t-2\tau\leq s\leq t} W^2(s).
\end{aligned}$$

It follows that,

$$\dot{W}_2(t) = -2(\beta - 2(N-1)\bar{g} - (N-1)\bar{g}\tau)W_2(t) + 4(N-1)\bar{g}\tau[\beta^2 + 2(N-1)^2\bar{g}^2] \sup_{t-2\tau\leq s\leq t} W_2(s).$$

From lemma 3,$W_2(t)$ is exponentially approach to zero,therefore,the synchronization state $S(t)$ of the delayed networks (1) is exponentially stable.

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