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Inverse Eigenvalue Problems and Their Associated Approximation Problems for Matrices with J-(Skew) Centrosymmetry

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Abstract The inverse problems play an important role in MEG reconstructions [3, 4, 5, 6, 7]. In this paper, a partially described inverse eigenvalue problem and an associated optimal approximation problem for J-centrosymmetric matrices are considered respectively. It is shown under which conditions the inverse eigenproblem has a solution. An expression of its general solution is given. In case a solution of the inverse eigenproblem exists, the optimal approximation problem can be solved. The formula of its unique solution is given. Also, the case for J-skew centrosymmetric matrices is considered.

Keywords J-centrosymmetric matrices; J-skew centrosymmetric; Inverse eigenvalue problem; optimal approximation

1 Introduction

A matrix $A \in \mathbb{C}^{2m\times 2m}$ is said to be: *J-centrosymmetric* if $AJ = JA$; *J-skew centrosymmetric* if $AJ = -JA$, where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, where I denotes the identity matrix of order m.

The symmetric skew Hamiltonian matrices occurring in mechanical and quantum mechanical problems form an important subclass of J centrosymmetric matrices and symmetric Hamiltonian matrices arising in solving continuous time linear quadratic optimal control problems, algebra Riccati equations form an important subclass of J skew centrosymmetric matrices, see for example [1] and references therein.

This paper focus on the inverse eigenvalue problems (IEPs) and the associated optimal approximations of J-(skew) centrosymmetric matrices.

Inverse eigenvalue problems have found some important applications in systems biology and bioinformatics [3, 4, 5, 6, 7]. An IEP concerns the reconstruction of a matrix from prescribed spectral data. To be more specific, given a set of k (not necessarily linearly independent) vectors $x_j \in \mathbb{F}^n, j = 1,\dots,k$ $(n > k)$ and a set of scalars $\lambda_j \in \mathbb{F}, j = 1,\dots,k$, find a matrix $A \in \mathbb{F}^{n\times n}$ such that

$$Ax_j = \lambda_j x_j \tag{1}$$

for $j = 1, \ldots, k$. Here $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ denotes the field of real or complex numbers. Usually, A is subject to additional constraints, typically given in the form that $A \in \Omega$ is required, where Ω denotes a certain subset of $n \times n$ matrices. Several different kinds of sets Ω have already been dealt with in the literature: Jacobi matrices, symmetric matrices, anti-symmetric matrices, anti-persymmetric matrices, unitary matrices, centro-symmetric matrices, (generalized) Toeplitz matrices, symmetric anti-bidiagonal matrices. This is by far not a complete list, see [2, 8, 12] for a recent review, a number of applications and an extensive list of references.

A problem closely related to the inverse eigenproblem (1) is the following optimal approximation problem: given a matrix $\widetilde{A} \in \mathbb{C}^{n\times n}$, find a matrix S with some prescribed spectral data that gives the best approximation to $\widetilde{A}$ in the Frobenius norm, that is,

$$||\widetilde{A} - S||_F = \inf_{A \in \mathscr{S}} ||\widetilde{A} - A||_F, \tag{2}$$

where $\mathscr{S}$ denotes the set of all possible solutions of (1). Such a problem may arise, e.g., when a preconditioner with a specific structure is sought in order to solve linear systems of equations efficiently, see e.g., [8]. If a structured inverse eigenproblem (1) is considered, that is, A is required to be in some set Σ, then we obtain a structured optimal approximation problem, where in addition to (2) $A \in \Omega$ is required.

This paper is organized as follows: after discussing the structure and properties of a J-(skew) centrosymmetric matrix, respectively, in next section, we then consider the inverse eigenvalue problems for such classes of matrices in Section 3. The optimal approximation problems are considered in Section 4 and a conclusion is given in last section.

2 Structure and properties

In this section we begin with some basic notation. Throughout this paper, let W^+ denote the Moore-Penrose inverse of W, let $\mathbb{C}_J^{2m\times 2m} = \{A \in \mathbb{C}^{2m\times 2m} | AJ = JA\}$, i.e., the set of all $2m \times 2m$ J-centrosymmetric matrices and $\mathbb{S}_J^{2m\times 2m} = \{A \in \mathbb{C}^{2m\times 2m} | AJ = -JA\}$, i.e., the set of all $2m \times 2m$ J-skew centrosymmetric matrices.

2.1 Structure

It is known that a matrix $A \in \mathbb{C}_J^{2m\times 2m}$ has the following structure

$$A = \begin{bmatrix} B & -C \\ C & B \end{bmatrix}, \quad B, C \in \mathbb{C}^{m\times m} \tag{3}$$

and a matrix $A \in \mathbb{S}_J^{2m\times 2m}$ has the following structure

$$A = \begin{bmatrix} \hat{B} & \hat{C} \\ \hat{C} & -\hat{B} \end{bmatrix}, \quad \hat{B}, \hat{C} \in \mathbb{C}^{m\times m}. \tag{4}$$

Let

$$P = \frac{\sqrt{2}}{2} \begin{bmatrix} I & I \\ iI & -iI \end{bmatrix}, \tag{5}$$

which is an unitary matrix. Then we have the following result, see [1, Theorem 2.5].

Lemma 1 *Let $A \in \mathbb{C}^{2m\times 2m}$, then*

(i) A matrix $A \in \mathbb{C}_J^{2m\times 2m}$ defined as in (3) if and only if

$$A = P \begin{bmatrix} M & \\ & N \end{bmatrix} P^H, \tag{6}$$

where $M = B - iC$ and $N = B + iC$.

(ii) A matrix $A \in \mathbb{S}_J^{2m\times 2m}$ defined as in (4) if and only if

$$A = P \begin{bmatrix} & \hat{M} \\ \hat{N} & \end{bmatrix} P^H, \tag{7}$$

where $\hat{M} = \hat{B} - i\hat{C}$ and $\hat{N} = \hat{B} + i\hat{C}$.

2.2 Properties

Here we discuss eigenstructures of $A \in \mathbb{C}_J^{2m\times 2m}$ and $A \in \mathbb{S}_J^{2m\times 2m}$, respectively.

Theorem 1 *Assume that $A \in \mathbb{C}^{2m\times 2m}$, and (λ, x) be a eigen pair of A.*

(i) If $A \in \mathbb{C}_J^{2m\times 2m}$, then $x \pm iJx$ is also an eigenvector corresponding to the eigenvalue λ. Furthermore, $iJ(x+iJx) = x+iJx$ and $iJ(x-iJx) = -(x-iJx)$.

(ii) If $A \in \mathbb{S}_J^{2m\times 2m}$, then $(-\lambda, Jx)$ is also an eigenpair of A.

Proof. From the hypothesis, we have that $A \in \mathbb{C}_J^{2m\times 2m}$, that is $AJ = JA$. Then $Ax = \lambda x$ implies $A(iJx) = \lambda(iJx)$ and $A(x \pm iJx) = \lambda(x \pm iJx)$ immediately. Thus we have shown that the first conclusion holds.

As for (ii), due to $AJ = -JA$ and $Ax = \lambda x$, we have $AJx = -\lambda Jx$ immediately. Thus we complete the proof. □

Partitioning $x \in \mathbb{C}^{2m}$ into the form $x^T = (x_1^T, x_2^T)$ with $x_i \in \mathbb{C}^m$, we have the following result.

Theorem 2 *Assume that $x \in \mathbb{C}^{2m}$ and P is defined as in (5). Then*

$$P^H(I - iJ)x = \sqrt{2}\begin{bmatrix} 0 \\ x_1 + ix_2 \end{bmatrix} \quad \text{and} \quad P^H(I + iJ)x = \sqrt{2}\begin{bmatrix} x_1 - ix_2 \\ 0 \end{bmatrix}.$$

Proof. A straightforward calculation gives the proof. □

3 Inverse eigenvalue problems

In this section we first deal with the inverse eigenvalue problem for J-centrosymmetric matrices. Due to special structure of eigenvectors of J-centrosymmetric matrices (Theorem 1 (i)), the IEP can be described as follows.

Problem I *Given $X = [x_1, x_2, \cdots, x_s] \in \mathbb{C}^{2m\times s}$, $Y = [y_1, y_2, \cdots, y_t] \in \mathbb{C}^{2m\times t}$, $\Lambda_1 = diag(\lambda_1, \lambda_2, \cdots, \lambda_s)$, and $\Lambda_2 = diag(\lambda_1, \lambda_2, \cdots, \lambda_t)$ with $s, t < m$, find an $2m\times 2m$ J-centrosymmetric matrix A such that*

$$A[X \quad Y] = [X \quad Y] \quad diag(\Lambda_1, \Lambda_2), \tag{8}$$

where X and Y are required to satisfy

$$iJX = X \quad \text{and} \quad iJY = -Y. \tag{9}$$

From Theorem 2, we have that

$$P^H[\begin{array}{cc} X & Y \end{array}] = \frac{\sqrt{2}}{2}\left[\begin{array}{cc} \hat{X} & 0 \\ 0 & \hat{Y} \end{array}\right], \quad \hat{X} \in \mathbb{C}^{m\times s}, \hat{Y} \in \mathbb{C}^{m\times t}. \tag{10}$$

By Lemma 1, the Problem I has a solution if and on if each of

$$M\hat{X} = \hat{X}\Lambda_1 \quad \text{and} \quad N\hat{Y} = \hat{Y}\Lambda_2 \tag{11}$$

has a solution.

Thus we can always reduce the structured inverse eigenproblem (8) into two smaller subproblems (11) with half size.

Theorem 3 *Assume that X, Y, Λ_1 and Λ_2 are given as in Problem I. Let $\hat{X}$ and $\hat{Y}$ be defined as in* (10). *Then Problem I has a solution if and only if*

$$\hat{X}\Lambda_1\hat{X}^+\hat{X} = \hat{X}\Lambda_1 \quad \text{and} \quad \hat{Y}\Lambda_2\hat{Y}^+\hat{Y} = \hat{Y}\Lambda_2. \tag{12}$$

Its general solution can be expressed as

$$A = P\left[\begin{array}{cc} \hat{X}\Lambda_1\hat{X}^+ + K_1(I_m - \hat{X}\hat{X}^+) & \\ & \hat{Y}\Lambda_2\hat{Y}^+ + K_2(I_m - \hat{Y}\hat{Y}^+) \end{array}\right]P^H,$$

where K_1, $K_2 \in \mathbb{C}^{m\times m}$.

Proof. From Lemma 1 and Theorem 1, it is sufficient to show $M\hat{X} = \hat{X}\Lambda_1$ if and only if $\hat{X}\Lambda_1\hat{X}^+\hat{X} = \hat{X}\Lambda_1$, and its general solution can be expressed as

$$M = \hat{X}\Lambda_1\hat{X}^+ + K_1(I_m - \hat{X}\hat{X}^+), \quad K_1 \in \mathbb{C}^{m\times m}, \tag{13}$$

and $N\hat{Y} = \hat{Y}\Lambda_2$ if and only if $\hat{Y}\Lambda_2\hat{Y}^+\hat{Y} = \hat{Y}\Lambda_2$, and its general solution can be expressed as

$$N = \hat{Y}\Lambda_2\hat{Y}^+ + K_2(I_m - \hat{Y}\hat{Y}^+) \quad K_2 \in \mathbb{C}^{m\times m}.$$

We prove the first equivalence; the proof of the second is similar.

If (12) holds, due to $(I_m - \hat{X}\hat{X}^+)\hat{X} = 0$, then we have

$$M\hat{X} = [\hat{X}\Lambda_1\hat{X}^+ + K_1(I_m - \hat{X}\hat{X}^+)]X = \hat{X}\Lambda_1, \tag{14}$$

which means (13) is its general solution. Conversely, if $M\hat{X} = \hat{X}\Lambda_1$ has a solution, then, (13) is a solution and (14) implies that $\hat{X}\Lambda_1\hat{X}^+\hat{X} = \hat{X}\Lambda_1$ holds. □

Please note, that the set of all possible solutions $\mathscr{S}$ to the problem I may be empty.

We now deal with the inverse eigenvalue problem for J-skew centrosymmetric matrices. Due to special structure of eigenvectors of J-skew centrosymmetric matrices (Theorem 1 (ii)), the IEP can be described as follows.

Problem II *Given $Z=[z_1,z_2,\cdots,z_s]\in\mathbb{C}^{2m\times s}$, $\Lambda=diag(\lambda_1,\lambda_2,\cdots,\lambda_s)$, with $s<m$, find an $2m\times 2m$ J-skew centrosymmetric matrix A such that*

$$AZ=Z\Lambda, \tag{15}$$

where we assume that $\lambda_i\neq-\lambda_j$.

Let $Z=Z_c+Z_s$, where $Z_c=\frac{Z+iJZ}{2}$ and $Z_s=\frac{Z-iJZ}{2}$. Then, by Theorem 2, we have

$$P^HZ_c=\begin{bmatrix} X\\ 0\end{bmatrix}\quad\text{and}\quad P^HZ_s=\begin{bmatrix} 0\\ Y\end{bmatrix} \tag{16}$$

By Lemma 1, the Problem II has a solution if and on if each of

$$\hat{M}Y=X\Lambda\quad\text{and}\quad\hat{N}X=Y\Lambda \tag{17}$$

has a solution.

Again, we can always reduce the structured inverse eigenproblem (15) into two smaller subproblems (17) with half size.

Theorem 4 *Assume that Z, Λ are given as in Problem II. Let X and Y be defined as in* (16). *Then Problem II has a solution if and only if*

$$X\Lambda Y^+Y=X\Lambda\quad\text{and}\quad Y\Lambda X^+X=Y\Lambda. \tag{18}$$

Its general solution can be expressed as

$$A=P\begin{bmatrix} & X\Lambda Y^++K_1(I_m-YY^+)\\ Y\Lambda X^++K_2(I_m-\hat{X}X^+) & \end{bmatrix}P^H,$$

where K_1, $K_2\in\mathbb{C}^{m\times m}$. Furthermore, in this case $(-\lambda_i,Jz_i)$, $i=1,\ldots,s$, are also eigenpairs of A.

Proof. Theorem 1 (ii) implies that the pairs $(-\lambda_i,Jz_i)$, $i=1,\ldots,s$, are eigenpairs of A if $AZ=Z\Lambda$. Therefore, by Lemma 1 and Theorem 1, it is sufficient to show $\hat{M}Y=X\Lambda$ if and only if $X\Lambda Y^+Y=X\Lambda$, its general solution can be expressed as

$$\hat{M}=X\Lambda Y^++K_1(I_m-YY^+); \tag{19}$$

and $\hat{N}X=Y\Lambda$ if and only if $Y\Lambda X^+X=Y\Lambda$, its general solution can be expressed as

$$\hat{N}=Y\Lambda X^++K_2(I_m-\hat{X}X^+).$$

We prove the first equivalence; the proof of the second is similar.

If (18) holds, due to $(I_m-YY^+)Y=0$, then we have

$$\hat{M}Y=[X\Lambda Y^++K_1(I_m-YY^+)]Y=X\Lambda, \tag{20}$$

which means (19) is its general solution. Conversely, if $\hat{M}Y=X\Lambda$ has a solution, then, (19) is a solution and (20) implies that $X\Lambda Y^+Y=X\Lambda$ holds. □

4 The best approximation problems

Here we will deal with the following structured optimal approximation problem: Given a matrix $\widetilde{A} \in \mathbb{C}^{2m\times 2m}$, find a matrix $S \in \mathscr{S}$ that gives the best approximation to $\widetilde{A}$ in the Frobenius norm, that is,

$$||\widetilde{A}-S||_F = \inf_{A\in\mathscr{S}} ||\widetilde{A}-A||_F, \tag{21}$$

where $\mathscr{S}$ denotes the set of all possible solutions of (8) or (15).

For any matrix $\widetilde{A}$, we have $\widetilde{A} = A_c + A_s$, where $A_c = \frac{1}{2}(\widetilde{A} + J\widetilde{A}J)$ and $A_s = \frac{1}{2}(\widetilde{A} - J\widetilde{A}J)$ are the projections of $\widetilde{A}$ on $C_J^{2m\times 2m}$ and $C_s^{2m\times 2m}$ with respect to the inner product $(F,G) = trace(G^H F)$, respectively. By Lemma 1,

$$A_c = P\begin{bmatrix} M_c & \\ & N_c \end{bmatrix} P^H \quad \text{and} \quad A_s = P\begin{bmatrix} & \hat{M}_s \\ \hat{N}_s & \end{bmatrix} P^H. \tag{22}$$

If $\mathscr{S} \subset \mathbb{C}_J^{2m\times 2m}$ is nonempty, we then have the following result.

Theorem 5 *Given $\widetilde{A} \in \mathbb{C}^{2m\times 2m}$. Under the assumptions of Theorem 3 and if $\mathscr{S}$ is nonempty, the problem (21) has a unique solution S, which can be expressed as*

$$S = P\begin{bmatrix} \hat{X}\Lambda_1\hat{X}^+ + M_c(I_m - \hat{X}\hat{X}^+) & \\ & \hat{Y}\Lambda_2\hat{Y}^+ + N_c(I_m - \hat{Y}\hat{Y}^+) \end{bmatrix} P^H, \tag{23}$$

where M_c and N_c are defined as in (22).

Proof. From the hypothesis, we have $\widetilde{A} \in \mathbb{C}^{n\times n}$ and $\widetilde{A} = A_c + A_s$. Using unitary invariance of F-norm, $I - XX^+ = (I - XX^+)^2 = (I - XX^+)^H$, $I - YY^+ = (I - YY^+)^2 = (I - YY^+)^H$, $M_c = M_cXX^+ + M_c(I - XX^+)$ and $N_c = N_cYY^+ + N_c(I - YY^+)$, we therefore have

$$\begin{aligned} ||\widetilde{A}-A||_F^2 = ||A_c - A||_F^2 + ||A_s||_F^2 = ||M_c - \hat{X}\Lambda_1\hat{X}^- K_1(I_m - \hat{X}\hat{X}^+)||_F^2 \\ + ||N_s - \hat{Y}\Lambda_2\hat{Y}^+ - K_2(I_m - \hat{Y}\hat{Y}^+)||_F^2 + ||\hat{M}_s||_F^2 + ||\hat{N}_s||_F^2 \\ = ||(M_cXX^+ - \hat{X}\Lambda_1\hat{X}^+)||_F^2 + ||(M_c - K_1)(I_m - \hat{X}\hat{X}^+)||_F^2 + ||N_sYY^+ - \hat{Y}\Lambda_2\hat{Y}^+||_F^2 + \\ ||(N_c - K_2)(I_m - \hat{Y}\hat{Y}^+)||_F^2 + ||\hat{M}_s||_F^2 + ||\hat{N}_s||_F^2. \end{aligned}$$

This implies $||\widetilde{A}-A||_F^2$ reaches it minimal if and only if

$$(M_c - K_1)(I_m - \hat{X}\hat{X}^+) = 0, \quad \text{and} \quad (N_c - K_2)(I_m - \hat{Y}\hat{Y}^+) = 0,$$

which means that (23) holds. In this case, $min||\widetilde{A}-A||_F^2 = ||(M_c\hat{X}\hat{X}^+ - \hat{X}\Lambda_1\hat{X}^+)||_F^2 + ||N_s\hat{Y}\hat{Y}^+ - \hat{Y}\Lambda_2\hat{Y}^+||_F^2 + ||\hat{M}_s||_F^2 + ||\hat{N}_s||_F^2$. □

If $\mathscr{S} \subset \mathbb{S}_J^{2m\times 2m}$ is nonempty, we have the following result.

Theorem 6 *Given $\widetilde{A} \in \mathbb{C}^{2m\times 2m}$. Under the assumptions of Theorem* 4 *and if $\mathscr{S}$ is nonempty, the problem (21) has a unique solution S, which can be expressed as*

$$A = P\begin{bmatrix} & X\Lambda Y^+ + \hat{M}_s(I_m - YY^+) \\ Y\Lambda X^+ + \hat{N}_s(I_m - \hat{X}X^+) & \end{bmatrix} P^H,$$

where $\hat{M}_s$ *and* $\hat{N}_s$ *are defined as in* (22).

Proof. Again, using the fact that for any matrix $\widetilde{A}$, $\widetilde{A} = A_c + A_s$, where $A_c = \frac{1}{2}(\widetilde{A} + J\widetilde{A}J)$ and $A_s = \frac{1}{2}(\widetilde{A} - J\widetilde{A}J)$ are the projections of $\widetilde{A}$ on $C_J^{2m\times 2m}$ and $C_s^{2m\times 2m}$ with respect to the inner product $(F,G) = trace(G^H F)$, respectively, and (22), we can complete the proof by a similar proof of Theorem 5. □

5 Conclusion

It is a basic tenet of numerical analysis that structure should be exploited whenever solving a problem. In numerical linear algebra, this translates into an expectation that algorithms for general matrix problems can be streamlined in the presence of properties such as symmetry, definiteness, sparsity, Hamiltonian, Toeplitz, Vandermonde, etc.. That is the so-called structured matrix problems.

There are many applications that generate structured matrices and by exploiting the structure one may be able to design faster and/or more accurate algorithms; furthermore, structure may also help in producing solutions which have more precise physical meaning [9, 10, 11].

Here we first exploit the special structure of matrices with J-centrosymmetry to propose an inverse eigenvalue problem and the associated optimal approximation problem for such a class of matrices, which may be of potential applications in bio-quantum mechanical problems. The conditions on which the IEP has a solution are discussed, and its general solution is given if it is solvable. For the associated optimal approximation problem, we show that there exist an unique solution if the set of solutions of IEP is not empty. The expression of this unique solution is presented.

The case for matrices with J-skew centrosymmetry is also discussed.

As we have showed, the core of this paper is to reduce each of the two problems under consideration into two smaller subproblems with half size, so that the structured solutions can be obtained. On the other hand, the structured algorithms for computing those problems can be easily developed, in which about half of the memory units and about fourth of computational costs are required, as compared to the standard approach for a arbitrary matrix.

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