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Valuation of Derivative on Asset with Network Price Externality Effects

Yoshio Tabata[1,*] Hiroyasu Akakabe[2]

[1]Graduate School of Business Administration, Nanzan University, Yamazato-cho 18, Showa-ku, Nagoya, Japan
[2]Faculty of Business Administration, Nanzan University, Japan

Abstract The present paper focuses on pricing the plain vanila option when the price process of underlying asset is described by the stochastic Verhulst-Gompertz Equation with network externality effects in a complete market. The method is based on the change of measure, Girsanov theorem and martingale valuation techinique. The application to an exchange option is made attempt and the valuation formula for this option like the Black-Scholes one is derived. A simple relation similar to the put-call-parity between the exchange call option and the put option is provided. Our results will be useful to analyze the price evolution at a sudden rise or crash of the stock market.

1 Introduction

Prices of some assets traded in a financial market have the tendency raised more when the prices begin to rise, it is because most traders expect that a price rises more and the purchase order of other traders is induced as a result. Conversely, when the prices begin to fall, selling orders of other traders are induced. In the actual market, we often observe such a phenomenon called *network externality price effects.*

The main purpose of this paper is devoted to find pricing formulae for a plain vanila and an exchange options when the price of the underlying asset shows the network externality price effects. In Section 2 we introduce the Verhulst-Gompertz equation to express the network externality effects and in Section 3 we develope the martingale measure to evaluate the value of option. In Section 4 we present a hedging strategy and a pricing formula of an exchange option that gives the holder the right which exchanges the underlying asset described by the stochastic Verhulst-Gompertz equation for the asset with the geometric Brownian motion (Black-Scholes model) by means of martingale pricing method. If we regard the former as the spot price of energy commodity and the latter as the stock price of business firm that trades the energy commodity, such an exchang option will be very useful to hedge the sudden rise or heavy fall in price of energy and to stabilize the economy.

2 Undelying Price Evolution

Consider a market consisting of one bond (riskless asset S^0) and two risky assets (S^1 and S^2) . The process $(S^1_t)_{0\leq t\leq T}$ denotes the price of the energy commodity such as crude

*E-mail: tabata@nanzan-u.ac.jp

oil and the process $(S_t^2)_{0\leq t\leq T}$ the stock price of energy business company at time t. The dynamics of S^0, S^1 and S^2 satisfy the stochastic differential equations:

$$\left.\begin{array}{ll} dS_t^0 = rS_t^0 dt, & S_0^0 = 1, \\ dS_t^1 = \mu_1 S_t^1 \Big[1-(\frac{S_t^1}{\beta})^\alpha\Big]dt + \sigma_1 S_t^1 w_t, & S_0^1 = s_1 > 0, \\ dS_t^2 = \mu_2 S_t^2 dt + \sigma_2 S_t^2 dB_t^2, & S_0^2 = s_2 > 0, \end{array}\right\} \tag{1}$$

with

$$w_t = \sqrt{1-\rho_{12}^2} B_t^1 + \rho_{12} B_t^2 \tag{2}$$

and the processes $B^1 = (B_t^1)$ and $B^2 = (B_t^2)$ are independent standard Brownian motions defined in a complete probability space $(\Omega, \mathscr{F}, \mathrm{P})$. Note that P is the physical probability measure, that captures the underlying uncertainty in this market.

Since $\mathbb{E}_{\mathrm{P}}[w_t] = 0$, $(B_t^1)_{0\leq t\leq T}$ and $(B_t^2)_{0\leq t\leq T}$ are mutually independent and $\mathrm{Var}(w_t) = 1$, the process w_t is a P-standard Brownian motion in the probability space $(\Omega, \mathscr{F}, \mathrm{P})$. Moreover, it is easily shown that the quadratic variation of w_t and the cross variation S_t^1 and S_t^2 are given by

$$d\langle w_\cdot, w_\cdot\rangle_t = dt, \quad d\langle S_\cdot^1, S_\cdot^2\rangle_t = \rho_{12}\sigma_1\sigma_2 S_t^1 S_t^2 dt.$$

Then, the second equation in (1) is regarded as an extension of the deterministic Verhulst-Gompertz equation to the stochastic version and S_t^1 depends on S_t^2. It is well known that the Verhulst-Gompertz equation has been extensively studied in biology and shows the growth of populations of individuals in some species.

3 Option Pricing by Martingale Measure

We consider a martingale method to evaluate the value of the plain vanila option with the price evolution of underlying asset described by the second equation in (1). To this end, we present the martingale measure common to two risky assets S^1 and S^2 and establish some preliminary lemmas for providing our main result.

Lemma 1 Let P-independent processes $(u_t^1)_{0\leq t\leq T}$ and $(u_t^1)_{0\leq t\leq T}$ be $\mathscr{L}^2$-integrable and satisfy the Novikov condition :

$$\mathbb{E}_{\mathrm{P}}\Big[\exp\big(\frac{1}{2}\int_0^t (u_s^k)^2 ds\big)\Big] < \infty.$$

Define the process $(M_t)_{0\leq t\leq T}$ by

$$M_t = \exp\Big(-\int_0^t u_s^1 dB_s^1 - \int_0^t u_s^2 dB_s^2 - \frac{1}{2}\int_0^t (u_s^1)^2 ds - \frac{1}{2}\int_0^t (u_s^1)^2 ds\Big).$$

If we define W_t^k, $(k = 1, 2)$ by

$$W_t^k = B_t^k + \int_0^t u_s^k ds, \quad (k = 1, 2),$$

then for any real number ξ, the complex valued process $(e^{\frac{\xi^2}{2}t - i\xi W_t^k} M_t^k)$ is a martingale relative to $(\mathscr{F}_t)_{0\leq t\leq T}$ under P, where $i = \sqrt{-1}$. □

The proof is immediately given by the application of Itô's formula to $(e^{\frac{\xi^2}{2}t-i\xi W_t^k}M_t^k)$. From this lemma, it follows that for $0 \le s \le t \le T$,

$$\frac{1}{M_s^k}\mathbb{E}_{\mathrm{P}}(e^{-i\xi(W_t^k-W_s^k)}M_t^k \mid \mathscr{F}_s) = e^{-\xi^2(t-s)/2}, \ \ (k=1,2). \tag{3}$$

Lemma 2 Assume that the same conditions in Lemma 1 are satisfied. Then, the process $(M_t)_{0\le t\le T}$ is a P-maritingale. Under the probability measure $\tilde{\mathrm{P}}$ equivalent to P defined by $d\tilde{\mathrm{P}} = M_T d\mathrm{P}$, the processes

$$W_t^k = B_t^k + \int_0^t u_s^k ds, \ \ 0 \le t \le T, \ \ (k=1,2)$$

are independent $\tilde{\mathrm{P}}$-standard Brownian motions and the vector process $(W_t^1 - W_s^1, W_t^2 - W_s^2)$ is independent of $\mathscr{F}_s$. □

Applying Itô's formula and by making the necessary substitutions we end up with following Lemma 3. Note that the lemma asserts the existence of the martingale measure $\tilde{\mathrm{P}}$, equivalent to P, and presents the construction method of such a measure $\tilde{\mathrm{P}}$.

Lemma 3 Define u_t^1 and u_t^2 by

$$u_t^1 \equiv \frac{1}{\sqrt{1-\rho_{12}^2}}\Big[\frac{\mu_1 - r}{\sigma_1} - \rho_{12}\frac{\mu_2 - r}{\sigma_2} - \frac{\mu_1}{\sigma_1}(\frac{S_t^1}{\beta})^\alpha\Big], \ \ u_t^2 = \frac{\mu_2 - r}{\sigma_2}. \tag{4}$$

If we denote

$$M_t^k = \exp\Big(-\int_0^t u_s^k dw_s - \frac{1}{2}\int_0^t (u_s^k)^2 ds\Big), \ \ k=1,2,$$

then the process $(M_t)_{0\le t\le T}$ defined by $M_t = M_t^1 M_t^2$ is a non-negative P-martngale. Furthermore, define the probability measure $\tilde{\mathrm{P}}$, equivalent to P, by

$$d\tilde{\mathrm{P}} = M_T d\mathrm{P}, \tag{5}$$

then the process $(W_t^k)_{0\le t\le T}$ defined by Girsanov's transformation

$$W_t^k \equiv B_t^k + \int_0^t u_s^k ds, \ \ k=1,2 \tag{6}$$

is a $\tilde{\mathrm{P}}$-martingale and the stochastic differential equation (1) is reduced to the geometric Brownian motion

$$dS_t^1 = rS_t^1 dt + \sigma_1 S_t^1 d\hat{W}_t, \tag{7}$$

where the process $(\hat{W}_t)_{0\le t\le T}$ is given by

$$\hat{W}_t = \sqrt{1-\rho_{12}^2}W_t^1 + \rho_{12}W_t^2. \quad \square$$

Consequently, as was proved in the standard textbook on finance, if there exists the probability measure $\tilde{\mathrm{P}}$, equivalent to P, which the discounted price process $\tilde{S}_t = e^{-rt}S_t$

becomes to be a martingale, then the European option is replicable and the value of the option is given by

$$V_t = \mathbb{E}_{\tilde{\mathrm{P}}}\big(e^{-r(T-t)}h\big|\mathscr{F}_t\big), \tag{8}$$

where h is $\mathscr{L}^2$-integrable $\mathscr{F}_T$-measurable payoff function at the maturity T. In our case, the discounted price process $\tilde{S}_t^1$ is a martingale since

$$\begin{aligned} d\tilde{S}_t^1 &= -r\tilde{S}_t^1 dt + \tilde{S}_t^1\big\{\mu_1\big[1-(\frac{S_t^1}{\beta})^\alpha\big]dt + \sigma\big(\sqrt{1-\rho_{12}^2}dB_t^1 + \rho_{12}dB_t^2\big)\big\} \\ &= -r\tilde{S}_t^1 dt + \tilde{S}_t^1[rdt + \sigma_1(\sqrt{1-\rho_{12}^2}\sigma_1 S_t^1 dW_t^1 + \rho_{12}\sigma_1 S_t^1 dW_t^2)] \\ &= \sigma_1\tilde{S}_t^1(\sqrt{1-\rho_{12}^2}\sigma_1 S_t^1 dW_t^1 + \rho_{12}\sigma_1 S_t^1 dW_t^2) = \sigma_1\tilde{S}_t^1 d\hat{W}_t. \end{aligned}$$

Therefore, these arguments establish that the option value with underlying asset S_t^1 satisfies the same type of Black-Scholes valuation formula. This implies that an application of martingale (risk-neutral) valuation gives the option price representation as

$$V_t = \mathbb{E}_{\tilde{\mathrm{P}}}(e^{-r(T-t)}h \mid \mathscr{F}_t) = S_t^1 N(d_1) + e^{-r(T-t)}KN(d_2), \tag{9}$$

where $h = (S_T^1 - K)_+$ and

$$d_1 = \frac{\log(S_t^1/K) + (r+\sigma_1^2/2)(T-t)}{\sigma_1\sqrt{T-t}}, \quad d_2 = d_1 - \sigma_1\sqrt{T-t}.$$

It follows that the option value is simply the expected payoff discounted at the risk-free rate, as should be the case with risk neutorality.

Using the law of iterated expectation, we obtain the following proposition about the option pricing, immediately.

Proposition 1 Assume that the price processes of asset 1 and asset 2 are given by the stochastic Verhulst-Gompertz equation and the Black-Scholes equation with same volatility. Let $C(t,S_t^1)$ and $C_{BS}(t,S_t^2)$ be the prices at time t, $(0 \le t \le T)$ of European style call (plain vanila) options with the same exercise price K and the maturity T on the underlying asset 1 and asset 2, respectively. If $S_s^1 = S_s^2$, ($\tilde{\mathrm{P}}$-$a.s.$) for some $s \in [0,T]$, then $\mathbb{E}_{\tilde{P}}[C(t,S_t^1)] = \mathbb{E}_{\tilde{P}}[C_{BS}(t,S_t^2)]$ for any $t \in [s,T]$.

4 Exchange Option and Hedging Strategy

An exchange option gives the holder the right to exchange one asset S^1 for another S^2. The payoff for this contract at maturity T is

$$(S_T^1 - S_T^2)_+ = \max(S_T^1 - S_T^2, 0).$$

Consider an investor with initial endowment $= V_0(\phi) \ge 0$ and investing in the assets stated above. Let H_t^0 be the number of riskless assets S^0, and H_t^1, H_t^2 be the number of risky assets S^1 and S^2, respectively, owned by the investor at time t. The triplet $\phi(t) = (H_t^0, H_t^1, H_t^2)$, $t \in [0,T]$ is called a *trading strategy* or a portfolio. We assume that H_t^0, H_t^1, H_t^2 are measurable and adapted processes such that

$$\int_0^T |H_t^0|dt + \int_0^T [(H_t^1)^2 + (H_t^2)^2]dt < \infty \quad \mathbb{P}-a.s.$$

Then $V_0(\phi) = H_0^0 + H_0^1 S_0^1 + H_0^2 S_0^2$ and the *value of the strategy* (the investor's wealth) at time t is

$$V_t(\phi) = H_t^0 S_t^0 + H_t^1 S_t^1 + H_t^2 S_t^2.$$

We say that the strategy ϕ is *self-financing* if there is no fresh investment and consumption. This means that

$$V_t(\phi) = V_0(\phi) + \int_0^t H_s^0 dS_s^0 + \int_0^t H_s^1 dS_s^1 + \int_0^t H_s^2 dS_s^2,$$

or,

$$dV_t(\phi) = H_t^0 dS_t^0 + H_t^1 dS_t^1 + H_t^2 dS_t^2.$$

Let $\tilde{S}$ and $\tilde{V}$ be the discounted processes. Then, the self-financing strategy ϕ is written by

$$\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t H_s^1 d\tilde{S}_s^1 + \int_0^t H_s^2 d\tilde{S}_s^2 \Rightarrow$$

$$d\tilde{V}_t = H_t^1 d\tilde{S}_t^1 + H_t^2 d\tilde{S}_t^2 = e^{-rt} H_t^1 \sigma_1 S_t^1 d\hat{W}_t + e^{-rt} H_t^2 \sigma_2 S_t^2 dW_t^2$$

Note that $\int_0^T H_t^i e^{-rt} S_t^i \sigma_i d\hat{W}_t^i, (i = 1,2)$ is a $\tilde{\mathbb{P}}$-martingale. So,

$$\tilde{V}_t = V_0 + \int_0^t e^{-rs} H_s^1 \sigma_1 S_s^1 d\hat{W}_s + \int_0^t e^{-rs} H_s^2 \sigma_2 S_s^2 dW_s^2$$

becomes to be a martingale. The argument described above leads to the following lemma:
Lemma 4 If the self-financing strategy ϕ is uniformly bounded and integrable :

$$\int_0^T |H_t^0| dt + \int_0^T (H_t^1)^2 dt + \int_0^T (H_t^2)^2 dt + < \infty \ (a.s.)$$

and

$$V_T = (S_T^1 - S_T^2)_+,$$

then for any $t \leq T$, the value of the strategy ϕ is given by

$$V_t = F(t, S_t^1, S_t^2),$$

where the function F is

$$F(t, x_1, x_2) = \mathbb{E}_{\tilde{\mathbb{P}}} \Big[\big(x_1 e^{\sigma_1 (\hat{W}_T - \hat{W}_t) - \sigma_1^2 (T-t)/2} - x_2 e^{\sigma_2 (W_T^2 - W_t^2) - \sigma_2^2 (T-t)/2} \big)_+ \Big].$$

Lemma 5 Suppose that g_1 and g_2 are independent random variables with standard normal distributions. Then for any real numbers $y_1, y_2, \lambda_0, \lambda_2$, the following relation holds:

$$\begin{aligned} \mathbb{E}\{[e^{y_1 + \lambda_1 g_1 + \lambda_2 g_2}]_+\} &= e^{y_1 + (\lambda_0^2 + \lambda_1^2)/2} N\Big(\frac{y_1 - y_2 + (\lambda_1^2 + \lambda_0^2) - \lambda_0 \lambda_2}{\sqrt{\lambda_1^2 + (\lambda_0 - \lambda_2)^2}} \Big) \\ &\quad - e^{y_2 + \lambda_2^2/2} N\Big(\frac{y_1 - y_2 - \lambda_2^2}{\sqrt{\lambda_1^2 + (\lambda_0 - \lambda_2)^2}} \Big). \quad \square \end{aligned} \tag{10}$$

As was pointed out in Lemma 2, two processes $\hat{W}_{T-t} = \hat{W}_T - \hat{W}_t$ and $W^2_{T-t} = W^2_T - W^2_t$ are independent standard Brownian motions. Hence, substituting

$$y_1 = \log x_1 - \frac{\sigma_1^2}{2}(T-t), \;\; \lambda_1 = \sigma_1 \sqrt{1-\rho_{12}^2}\sqrt{T-t}, \;\; \lambda_0 = \sigma_1 \rho_{12}\sqrt{T-t},$$

$$y_2 = \log x_2 - \frac{\sigma_2^2}{2}(T-t), \;\; \lambda_2 = \sigma_2 \sqrt{T-t}$$

into the equation (10) in Lemma 5, we get the explicit value form of exchange option $F = F(t, x_1, x_2)$ as

$$F = x_1 N\Big(\frac{\log(x_1/x_2) + \frac{1}{2}D^2(T-t)}{D\sqrt{T-t}}\Big) - x_2 N\Big(\frac{\log(x_1/x_2) - \frac{1}{2}D^2(T-t)}{D\sqrt{T-t}}\Big),$$

where $D = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$.

To derive the replicated strategy, let $C_t = F(t, S^1_t, S^2_t)$, then the discounted value of exchange option is given by $\tilde{C}_t = F(t, \tilde{S}^1_t, \tilde{S}^2_t)$. By Itô's formula with

$$d\tilde{S}^1_t = \sigma_1^2 \tilde{S}^1_t d\tilde{W}_t, \;\; d\tilde{S}^2_t = \sigma_2^2 \tilde{S}^2_t d\tilde{W}_t,$$

we obtain

$$\tilde{C}_t = F(0, \tilde{S}^1_0, \tilde{S}^2_0) + \int_0^t \frac{\partial F}{\partial x_1}(u, \tilde{S}^1_u, \tilde{S}^2_u) d\tilde{S}^1_u + \int_0^t \frac{\partial F}{\partial x_2}(u, \tilde{S}^1_u, \tilde{S}^2_u) d\tilde{S}^2_u. \quad (11)$$

Thus,

$$H^0_t = F(t, \tilde{S}^1_t, \tilde{S}^2_t) - \frac{\partial F}{\partial x_1}(t, \tilde{S}^1_t, \tilde{S}^2_t)\tilde{S}^1_t - \frac{\partial F}{\partial x_2}(t, \tilde{S}^1_t, \tilde{S}^2_t)\tilde{S}^2_t$$

$$H^1_t = \frac{\partial F}{\partial x_1}(t, \tilde{S}^1_t, \tilde{S}^2_t), \;\; H^2_t = \frac{\partial F}{\partial x_1}(t, \tilde{S}^1_t, \tilde{S}^2_t).$$

Also, we obtain $\partial F/\partial x_2$. Hence, we find that the strategy (H^0, H^1, H^2) which satisfies

$$\partial F(t, \tilde{S}^1_t, \tilde{S}^2_t)/\partial x_k = \partial F(t, S^1_t, S^2_t)/\partial x_k, \;\; k = 1, 2$$

is self-financing. The hedging of option is the problem faced by a financial institution that sells to a client some contract designed to reduce the client's risk.

From these lemmas and propositions, we have main theorem as

Theorem Assume that the price S^1_t of asset 1 with network externality effect and the price S^2_t of asset 2 are given by equation (1). The price C_t of exchange option that exchanges the 1 unit of asset to the 1 unit of asset 2 at maturity T is given by

$$C_t = F(t, S^1_t, S^2_t),$$

where the function F is

$$F(t, S^1_t, S^2_t) = x_1 N(d'_1) - x_2 N(d'_2), \quad (12)$$

where $d_1' = \frac{\log(x_1/x_2)+\frac{1}{2}D^2(T-t)}{D\sqrt{T-t}}$ and $d_2' = d_1' - \sqrt{T-t}(D + \frac{\rho_{12}\sigma_1\sigma_2}{D})$. This exchange option is relicated by the self-financing strategy $\phi = (H^0, H^1, H^2)$, where H_t^k, $(k = 0, 1, 2)$ are given by

$$H_t^0 = e^{-rt}\left(F(t, S_t^1, S_t^2) - \frac{\partial F}{\partial x_1}(t, S_t^1, S_t^2)S_t^1 - \frac{\partial F}{\partial x_2}(t, S_t^1, S_t^2)S_t^2\right)$$

$$H_t^1 = \frac{\partial F}{\partial x_1}(t, S_t^1, S_t^2), \ \ H_t^2 = \frac{\partial F}{\partial x_2}(t, S_t^1, S_t^2) \quad \square$$

Using the relation

$$(S_t - K)_+ - (S_t - K)_- = S_t - K$$

deduces an important link between the process of call and put options which corresponds to the put-call-parity for European options.

Proposition 4 Let C_t be the price of exchange option and P_t be the price of exchange option which is symetric to C_t, that is, the holder of the option P_t has the right, but not the obligation, to exchange an underlying asset at maturity T for a contractually. Then, for any t $(0 \leq t \leq T)$,

$$C_t - P_t = S_t^1 - S_t^2 \quad \square$$

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