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A Full-Newton Step Infeasible Interior-Point Algorithm for Linear Programming Based on a Special Self-Regular Proximity*

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Abstract This paper proposes an infeasible interior-point algorithm with full-Newton step for linear programming, which is an extension of the work of Roos (SIAM J. Optim., 16(4):1110–1136, 2006). We introduce a special self-regular proximity to induce the feasibility step and to verify quadratic convergence. The result of polynomial complexity coincides with the best-known iteration bound for infeasible interior-point methods, namely, $O(n\log n/\varepsilon)$.

Keywords Linear programming; Infeasible interior-point methods; Full-Newton step; Polynomial complexity; Self-regular proximity

1 Introduction

We are concerned with the (LP) problem given in the following standard form:

$$(P) \qquad \begin{aligned} &\min \ c^T x \\ &s.t. \ \ Ax = b, \quad x \geq 0, \end{aligned}$$

and its associated dual problem:

$$(D) \qquad \begin{aligned} &\max \ b^T y \\ &s.t. \ \ A^T y + s = c, \quad s \geq 0, \end{aligned}$$

where $c, x, s \in \mathbb{R}^n$, $b, y \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ is of full row rank.

For a comprehensive learning about interior-point methods (IPMs), we refer to Roos *et al.* [5]. In Roos [6], a full-Newton step infeasible interior-point algorithm for linear programming (LP) was presented and he also proved that the complexity of the algorithm coincides with the best known iteration bound for infeasible IPMs. In Liu and Sun [1], Mansouri and Roos [2], they defined the feasibility step by special search directions, respectively. Such directions can be seen as parameterized affine scaling directions.

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Recently Peng *et al.* [3] introduced a new class of primal-dual IPMs based on self-regular proximities. These methods do not use the classic Newton directions. Instead they use a direction that can be characterized as a steepest descent direction (in a scaled space) for a so-called self-regular barrier function. Each such barrier function is determined by a simple univariate self-regular function, called its kernel function. Salahi [7] extended the method in Peng and Terlaky [4] to infeasible IPMs. For both of them, the center path neighborhood are defined by the proximity function and they don't utilize any inner iteration to get centered.

Inspired by Salahi [7], we develop an infeasible IPMs with full-Newton steps for (LP). The search direction of the feasibility step is induced by the proximity function. The feasibility step of the latter is due to the classic primal-dual Newton direction, which is induced by the logarithmic barrier function. And the proximity function is used to verify the quadratic convergence in this paper. We also use a norm-based proximity to define the central neighborhood. Similar to the process of the analysis, we can get the same result of polynomial complexity, that is, $n\log n/\varepsilon$, which is the best currently for infeasible IPMs.

Throughout the paper $\|\cdot\|$ denotes the l_2-norm. We use Φ to denote the proximity function though $\Phi(v)$ and $\Phi(x,s;\mu)$ have different domains.

2 The statement of algorithm

As usual for infeasible IPMs we assume that the initial iterates (x^0,y^0,s^0) are as follows:

$$x^0 = s^0 = \zeta e, \quad y^0 = 0, \quad \mu^0 = \zeta^2,$$

where e is the all-one vector of length n, μ^0 is the initial dual gap and $\zeta > 0$ is such that

$$\|x^* + s^*\|_\infty \leq \zeta,$$

for some optimal solution (x^*,y^*,s^*) of (P) and (D).

After defining r_b^0 and r_c^0 as the initial residual vectors:

$$\begin{aligned} r_b^0 &= b - Ax^0, \\ r_c^0 &= c - A^T y^0 - s^0, \end{aligned}$$

we recall the main ideas underlying the algorithm in Roos [6]. For any ν with $0 < \nu \leq 1$ we consider the perturbed problem (P_ν), defined by

$$(P_\nu) \qquad \min\{(c - \nu r_c^0)^T x : \ Ax = b - \nu r_b^0, \ x \geq 0\},$$

and its dual problem (D_ν), which is given by

$$(D_\nu) \qquad \max\{(b - \nu r_b^0)^T y : \ A^T y + s = c - \nu r_c^0, \ s \geq 0\}.$$

Note that if $\nu = 1$ then $x = x^0$ yields a strictly feasible solution of (P_ν), and $(y,s) = (y^0,s^0)$ a strictly feasible solution of (D_ν). Due to the choice of the initial iterates we may conclude that if $\nu = 1$ then (P_ν) and (D_ν) each have a strictly feasible solution, which means that both perturbed problems then satisfy the well known interior-point condition (IPC).

Lemma 1. ([6, Lemma 1.1]) *The perturbed problems (P_ν) and (D_ν) satisfy the IPC for each $\nu \in (0,1]$, if and only if the original problems (P) and (D) are feasible.*

Assuming that (P) and (D) are feasible, it follows from Lemma 1 that the problems (P_ν) and (D_ν) satisfy the IPC, for each $\nu \in (0,1]$. And then their central paths exist. This means that the system

$$\begin{aligned} b - Ax &= \nu r_b^0, \quad x \geq 0, && (1)\\ c - A^T y - s &= \nu r_c^0, \quad s \geq 0, && (2)\\ xs &= \mu e && (3) \end{aligned}$$

has a unique solution for every $\mu > 0$, where xs denotes a Hadamard (componentwise) product of two vectors x and s. If $\nu \in (0,1]$ and $\mu = \nu\zeta^2$ we denote this unique solution in the sequel as $(x(\nu), y(\nu), s(\nu))$. As a consequence, $x(\nu)$ is the μ-center of (P_ν) and $(y(\nu), s(\nu))$ the μ-center of (D_ν). Due to this notation we have, by taking $\nu = 1$,

$$(x(1), y(1), s(1)) = (x^0, y^0, s^0) = (\zeta e, 0, \zeta e).$$

One measures proximity of iterates (x,y,s) to the μ-center of the perturbed problems (P_ν) and (D_ν) by the quantity $\delta(x,s;\mu)$, which is defined as follows:

$$\delta(x,s;\mu) := \delta(v) := \frac{1}{2}\|v - v^{-1}\|, \qquad \text{where} \qquad v := \sqrt{\frac{xs}{\mu}}. \tag{4}$$

Initially one has $x = s = \zeta e$ and $\mu = \zeta^2$, whence $v = e$ and $\delta(x,s;\mu) = 0$. In the sequel assuming that at the start of each iteration, $\delta(x,s;\mu)$ is smaller than or equal to a (small) threshold value $\tau > 0$. So this is certainly true at the start of the first iteration.

For the feasibility step in Roos [6] they used search directions $\Delta^f x$, $\Delta^f y$ and $\Delta^f s$ that are (uniquely) defined by the system

$$\begin{aligned} A\Delta^f x &= \theta \nu r_b^0, && (5)\\ A^T\Delta^f y + \Delta^f s &= \theta \nu r_c^0, && (6)\\ s\Delta^f x + x\Delta^f s &= \mu e - xs. && (7) \end{aligned}$$

In the centering steps, starting at the iterates $(x,y,s) = (x^f, y^f, s^f)$ and targeting at the μ-centers, the search directions $\Delta x, \Delta y, \Delta s$ are the usual primal-dual Newton directions, (uniquely) defined by

$$\begin{aligned} A\Delta x &= 0,\\ A^T\Delta y + \Delta s &= 0,\\ s\Delta x + x\Delta s &= \mu e - xs. \end{aligned}$$

Denoting the iterates after a centering step as x^+, y^+ and s^+, we recall the following results from Roos [5].

Lemma 2. *If $\delta := \delta(x,s;\mu) \leq 1$, then the primal-dual Newton step is feasible, i.e., x^+ and s^+ are nonnegative, and $(x^+)^T s^+ = n\mu$. Moreover, if $\delta := \delta(x,s;\mu) \leq 1/\sqrt{2}$, then $\delta(x^+, s^+; \mu) \leq \delta^2$.*

The centering steps serve to get iterates that satisfy $x^T s = n\mu^+$ and $\delta := \delta(x,s;\mu) \leq \tau$, where τ is (much) smaller than $1/\sqrt{2}$. By using Lemma 2, the required number of centering steps can easily be obtained. Because after the μ-update we have $\delta = \delta(x^f, s^f; \mu^+) \leq 1/\sqrt{2}$, and hence after k centering steps the iterates (x,y,s) satisfy

$$\delta(x,s;\mu^+) \leq (\frac{1}{\sqrt{2}})^{2^k}.$$

From this one easily deduces that no more than

$$\log_2(\log_2 \frac{1}{\tau^2}) \tag{8}$$

centering steps are needed.

Defining

$$d_x^f := \frac{v\Delta^f x}{x}, \quad d_s^f := \frac{v\Delta^f s}{s}, \tag{9}$$

with v as defined in (4). The system which defines the search directions $\Delta^f x$, $\Delta^f y$ and $\Delta^f s$, can be expressed in terms of the scaled search directions d_x^f and d_s^f as follows:

$$\begin{aligned} \bar{A}d_x^f &= \theta v r_b^0, \\ \bar{A}^T \frac{\Delta^f y}{\mu} + d_s^f &= \theta v v s^{-1} r_c^0, \\ d_x^f + d_s^f &= v^{-1} - v, \end{aligned}$$

where

$$\bar{A} = AV^{-1}X, \quad V = \mathrm{diag}(v), \quad X = \mathrm{diag}(x).$$

Note that the right-hand side of the third equation in the system is the negative gradient induced by the logarithmic barrier function

$$\Psi(v) := \sum_{i=1}^{n} \psi(v_i), \quad v_i = \sqrt{\frac{x_i s_i}{\mu}},$$

whose kernel function is

$$\psi(t) = \frac{1}{2}(t^2 - 1) - \log t.$$

In this paper the feasibility step is a slight modification of the classic primal-dual Newton direction. The feasibility direction is defined by a new system as follows

$$\begin{aligned} \bar{A}d_x^f &= \theta v r_b^0, \\ \bar{A}^T \frac{\Delta^f y}{\mu} + d_s^f &= \theta v v s^{-1} r_c^0, \\ d_x^f + d_s^f &= -\nabla\Phi(v), \end{aligned}$$

where $\Phi(v)$ is

$$\Phi(v) := \sum_{i=1}^{n} \phi(v_i),$$

and the kernel function of $\Phi(v)$ is defined as

$$\phi(t) := \frac{1}{2}(t - \frac{1}{t})^2.$$

Since $\phi'(t) = t - 1/t^3$, the third equation in the system can be written as

$$d_x^f + d_s^f = v^{-3} - v. \tag{10}$$

The next lemma focus on the effect of the feasible search direction induced by the self-regular proximity function.

Lemma 3. *If $\Phi(v) := \Phi(x,s;\mu) \leq 2$, then the primal-dual Newton step is feasible, i.e., x^+ and s^+ are nonnegative, and $(x^+)^T s^+ = n\mu$. Moreover, if $\Phi(v) := \Phi(x,s;\mu) \leq 1$, then $\Phi(x^+,s^+;\mu) \leq (\frac{1}{\sqrt{2}}\Phi(v))^2$.*

The following lemma quantifies the effect on the proximity measure if v is replaced by $\tilde{v} = \sqrt{1-\theta}v$.

Lemma 4. *Let (x,s) be a positive primal-dual pair and $\mu > 0$ such that $x^T s = n\mu$. Moreover let $\Phi(v) = \Phi(x,s;\mu)$ and $\tilde{v} := \sqrt{1-\theta}v$. Then*

$$\Phi(\tilde{v}) = \frac{1}{1-\theta}\Phi(v) + \frac{\theta^2 n}{1-\theta}.$$

3 Main Result

Defining

$$\rho(\Phi(v)) := (\Phi(v)+1) + \sqrt{(\Phi(v)+1)^2 - 1},$$

and

$$\omega_i := \omega_i(v) := \frac{1}{2}\sqrt{|d_{xi}^f|^2 + |d_{si}^f|^2},$$

and

$$\omega := \omega(v) := \|(\omega_1, \ldots, \omega_n)\|,$$

one can get the following result.

Lemma 5. *Assuming $v^{-2} + d_x^f d_s^f > 0$, one has*

$$2\Phi(v^f) \leq \frac{2}{1-\theta}\Phi(v) + \frac{\theta^2 n}{1-\theta} + \frac{2\omega^2}{1-\theta} + \frac{2(1-\theta)\rho(\Phi(v))^4\omega^2}{1-2\rho(\Phi(v))^2\omega^2}.$$

At this stage we decide to choose

$$\tau = \frac{1}{4}, \quad \theta = \frac{\alpha}{4\sqrt{n}}, \quad \alpha = \frac{1}{20\sqrt{n}}, \tag{11}$$

one can verify that

$$\omega \leq \frac{1}{2\sqrt{2}} \quad \Rightarrow \quad \Phi(v^f) \leq 1. \tag{12}$$

Finally we can get the following result of polynomial complexity.

Lemma 6. *The total number of inner iterations is bounded above by*

$$320n \log \frac{\max\{n\zeta^2, \|r_b^0\|, \|r_c^0\|\}}{\varepsilon}.$$

4 Concluding remarks

In this paper we introduce a self-regular proximity in the infeasible interior-point algorithm with full-Newton step for linear programming. We also use a norm-based proximity to define the central neighborhood. We only discuss a special self-regular proximity in this paper, our future work will focus on more general self-regular proximities.

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