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Equivalence between Generalized Saddle Point and Sun-point*

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Abstract In this paper concept of generalized saddle point(GSP) is introduced and applied to an optimization problem of a set of functionals on a Banach space, which shows that GSP and the optimum solution are equivalent. Although sun-point is a essential concept in approximation theory, their relation is seldom discussed. In this paper sun-point is also introduced to depict the optimum solution of a problem, and if functionals are convex, the equivalence between GSP and sun-point is established.

Keywords Generalized saddle point; Banach space; Sun-point; Nonlinear Approximation; Optimization

1 Introduction

Saddle point has been much interested by people because of its wide range of application in mathematics, engineering and game theory[4, 5, 8]. The dominance of saddle point is that it represents an optimum solution of a 'Min-Max' problem without using differentiability of functions. The book written by Eberhard Zeidler shows that it is very essential and useful in optimization theories[5]. The concept of generalized saddle point (GSP) has been introduced in [1] and is used to study the integrated optimization problem of a set of convex functionals on a *Banach* space. Extensive investigation of GSP method in approximation theory has been done in [2, 3]. Sun-point is an important concept in nonlinear best approximation theory, which is the generalization of convex set. However, Sun-point in best approximation theory is different from the extreme point of convex set. The aim of this paper is to establish the equivalence between GSP and sun-point, which is seldom discussed in the existed papers.

Assume X is a Banach space and H is a set of continuous and real-valued functions φs on X, i.e., $\varphi : X \to F$, where F is the real number field. We define $\Gamma = \sup_{\varphi\in H} \varphi$, that is $\Gamma(x) = \sup_{\varphi\in H} \varphi(x)$, $x \in X$. Let G be a subset of X, we consider the following optimization

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problem

$$(\Gamma,G): \qquad \inf_{g\in G}\Gamma(g).$$

If $g_0 \in G$, it satisfies $\Gamma(g_0)=\inf_{g\in G}\Gamma(g)$, then we call g_0 be an optimum solution of (Γ,G). Let $P_{(\Gamma,G)}$ be the set of all the optimum solution of (Γ,G), namely,

$$P_{(\Gamma,G)}=\{g_0\in G: \quad \Gamma(g_0)=\inf_{g\in G}\Gamma(g)\}.$$

Let φ be in H, the notations of $\inf_{g\in G}\varphi(g)$, $P_{(\varphi,G)}$, $\varphi(g_0)=\inf_{g\in G}\varphi(g)$ have the similar meaning as the above $\inf_{g\in G}\Gamma(g), P_{(\Gamma,G)}, \Gamma(g_0)$ respectively.

The rest of this paper proceeds as follows. In section 2 we introduce the concept of generalized saddle point and discuss the equivalent relation with optimum solution. Section 3 presents equivalence between GSP and sun-point.

2 Generalized saddle point solution of (Γ,G)

Let X be a Banach space and $\tilde{H}$ be a set of the real-valued functions on X. Now we define a function $\Psi:(\tilde{H},X)\to R$, that is

$$\Psi(\varphi,x)=\varphi(x), \qquad \forall(\varphi,x)\in(\tilde{H},X).$$

Let H be a subset of $\tilde{H}$ and G be the subset of X, we also define

$$\Gamma(x)=\sup_{\varphi\in H}\varphi(x)=\sup_{\varphi\in H}\Psi(\varphi,x),$$

then we define an optimization problem (Γ,G)

$$(\Gamma,G): \quad \inf_{g\in G}\Gamma(g)=\inf_{g\in G}\sup_{\varphi\in H}\Psi(\varphi,g).$$

Now we introduce the following set for convenient discussion,

$$M_{(\Gamma,x)}=\{\varphi_0\in H: \quad \Gamma(x)=\varphi_0(x), \quad x\in X\}.$$

Definition 1. Let $(\bar{\varphi},\bar{g})\in(H,G)$, we call $(\bar{\varphi},\bar{g})$ to be generalized saddle point (GSP) of Ψ in (H,G), if it satisfies the following condition,

$$\Psi(\varphi,\bar{g})\le\Psi(\bar{\varphi},\bar{g})\le\Psi(\bar{\varphi},g), \qquad (\varphi,g)\in(H,G).$$

The notion of saddle point is a fundamental concept in many areas of science and economics. A classical instance is the famous saddle point theorem for a zero-sum matrix game due to J. Von Neumann and O. Morgenstern [8].

Lemma 1. *Let $(\bar{\varphi},\bar{g})\in(H,G)$ be the GSP of Ψ, then $\bar{g}\in P_{(\Gamma,G)}$ and $\bar{\varphi}\in M_{(\Gamma,\bar{g})}$.*

Proof. Let $(\bar{\varphi},\bar{g}) \in (H,G)$ be the GSP of Ψ, then we have $\Psi(\varphi,\bar{g}) \leq \Psi(\bar{\varphi},\bar{g})$, that is $\varphi(\bar{g}) \leq \bar{\varphi}(\bar{g})$. Then we know that $\sup\limits_{\varphi\in H} \varphi(\bar{g}) \leq \bar{\varphi}(\bar{g})$, that is to say $\Gamma(\bar{g}) \leq \bar{\varphi}(\bar{g})$. Conversely, it is easy to know $\Gamma(\bar{g}) \geq \bar{\varphi}(\bar{g})$. Hence, we get

$$\Gamma(\bar{g}) = \bar{\varphi}(\bar{g}), \tag{1}$$

which means $\bar{\varphi} \in M_{(\Gamma,\bar{g})}$. On the other hand, by the inequality $\Psi(\bar{\varphi},\bar{g}) \leq \Psi(\bar{\varphi},g)$, we get

$$\bar{\varphi}(\bar{g}) \leq \inf_{g\in G} \bar{\varphi}(g) \leq \inf_{g\in G} \sup_{\varphi\in H} \varphi(g) = \inf_{g\in G} \Gamma(g). \tag{2}$$

Therefor we have $\Gamma(\bar{g}) \leq \inf\limits_{g\in G} \Gamma(g)$ from inequalities (1) and (2), that is to say, $\bar{g} \in P_{(\Gamma,G)}$. □

Lemma 2. *Let $\bar{\varphi} \in H, \bar{g} \in P_{(\bar{\varphi},G)}$ and $\bar{\varphi} \in M_{(\Gamma,\bar{g})}$, then $\bar{g} \in P_{(\Gamma,G)}$.*

Proof. Let $\bar{\varphi} \in H, \bar{g} \in P_{(\bar{\varphi},G)}$ and $\bar{\varphi}(\bar{g}) = \Gamma(\bar{g})$. Then, for any g in G, we have

$$\Gamma(\bar{g}) = \bar{\varphi}(\bar{g}) = \inf_{g\in G} \bar{\varphi}(g) \leq \bar{\varphi}(g) \leq \Gamma(g).$$

So we have $\Gamma(\bar{g}) \leq \inf\limits_{g\in G} \Gamma(g)$. It is evident to get the inequality $\inf\limits_{g\in G} \Gamma(g) \leq \Gamma(\bar{g})$. Hence, we get the equality $\Gamma(\bar{g}) = \inf\limits_{g\in G} \Gamma(g)$, i.e., $\bar{g} \in P_{(\Gamma,G)}$. □

Lemma 3. *Let $(\bar{\varphi},\bar{g}) \in (H,G)$, if $\bar{g} \in P_{(\bar{\varphi},G)}$, $\bar{\varphi}$ in $M_{(\Gamma,\bar{g})}$, then $(\bar{\varphi},\bar{g})$ is a GSP of Ψ in (H,G).*

Proof. Let $\bar{g} \in P_{(\bar{\varphi},G)}, \bar{\varphi} \in M_{(\Gamma,\bar{g})}$, then using the Lemma 2 we have $\bar{g} \in P_{(\Gamma,G)}$. Furthermore, we get

$$\sup_{\varphi\in H} \Psi(\varphi,\bar{g}) = \Gamma(\bar{g}) = \bar{\varphi}(\bar{g}) = \Psi(\bar{\varphi},\bar{g}). \tag{3}$$

Therefor we have the inequality $\Psi(\varphi,\bar{g}) \leq \Psi(\bar{\varphi},\bar{g})$ for any φ in H.

On the other hand, for arbitrary $g \in G$, we get

$$\Psi(\bar{\varphi},\bar{g}) = \Gamma(\bar{g}) = \bar{\varphi}(\bar{g}) \leq \bar{\varphi}(g) = \Psi(\bar{\varphi},g), \tag{4}$$

by using $\bar{g} \in P_{(\bar{\varphi},G)}$. Then the following inequalities are established

$$\Psi(\varphi,\bar{g}) \leq \Psi(\bar{\varphi},\bar{g}) \leq \Psi(\bar{\varphi},g) \qquad \forall(\varphi,g) \in (H,G)$$

by using (3) and (4), which implies that $(\bar{\varphi},\bar{g})$ is the GSP of Ψ in (H,G). □

We are able to get the following theorem by lemma 1 and 3,

Theorem 4. *Let $(\bar{\varphi},\bar{g}) \in (H,G)$, then $\bar{g} \in P_{(\bar{\varphi},G)}$ and $\bar{\varphi} \in M_{(\Gamma,\bar{g})}$, if and only if $(\bar{\varphi},\bar{g})$ is a GSP of Ψ in (H,G).*

3 Equivalence Between GSP and sun-point

We introduce the following notations for convenient discussion. Directional derivative of function φ and Γ at x (see [7] p.16) are defined respectively as

$$\varphi'(x,y) = \lim_{\alpha\to 0^+} \frac{\varphi(x+\alpha y)-\varphi(x)}{\alpha}, \qquad \varphi \in H,$$

$$\Gamma'(x,y) = \lim_{\alpha\to 0^+} \frac{\Gamma(x+\alpha y)-\Gamma(x)}{\alpha},$$

and $\tilde{G}_{g_0}$ is defined as following,

$$\tilde{G}_{g_0} = \bigcup_{g\in G}\{g_\alpha : \quad g_\alpha = (1-\alpha)g_0 + \alpha g, \quad \alpha\in[0,1], g_0 \in G\},$$

Definition 2. We say that g_0 is a sun-point (see [6]) of Γ in G , if $g_0 \in P_{(\Gamma,G)}$ implies $g_0 \in P_{(\Gamma,\tilde{G}_{g_0})}$, where $P_{(\Gamma,\tilde{G}_{g_0})}$ is the optimum solution set of $(\Gamma,\tilde{G}_{g_0})$.

Definition 3. Let $x \in X$, and $\|x\|_X < \infty$, we say H be bounded with respect to x, if there is a positive number $M > 0$ such that $|\varphi(x)| \le M$ for all $\varphi \in H$, where $\|\cdot\|_X$ is the norm of Banach space X.

Definition 4. We say H is a closed set with respect to x, if the numeral set $\{\varphi(x)|\varphi \in H, \|x\|_X < \infty, x \in X\}$ is a closed set.

Lemma 5. *Let x be in X with $\|x\|_X < \infty$, if H is a closed and bounded set with respect to x, then $M_{(\Gamma,x)}$ is a nonempty set.*

Proof. We assume that $M_{(\Gamma,x)}$ is an empty set with $\|x\|_X < \infty, x\in X$, i.e., for any $\varphi \in H$, we have $\varphi(x) < \Gamma(x)$.
By $\Gamma(x) = \sup_{\varphi\in H}\varphi(x)$, we obtain that there exists a sequence $\{\varphi_i\}_{i=1}^{\infty} \subset H$ such that

$$\lim_i \varphi_i(x) = \Gamma(x).$$

By the set $\{\varphi(x)|\varphi\in H, \|x\|_X < \infty, x\in X\}$ being a closed and bounded set, we get that there exists a convergent subsequence $\{\varphi_{i_k}\}_{k=1}^{\infty}$ of $\{\varphi_i\}_{i=1}^{\infty}$ and a $\varphi_0 \in H$ such that

$$\Gamma(x) = \lim_k \varphi_{i_k}(x) = \varphi_0(x).$$

By the assumption at the beginning of the proof, we get the contradiction

$$\Gamma(x) < \Gamma(x),$$

which shows that the set $M_{(\Gamma,x)}$ is not empty. □

This theorem represents the rationality of the set $M_{(\Gamma,x)}$.

Lemma 6. *Let $x,y \in X$, H be closed and bounded with respect to x, then we have $\Gamma'(x,y) = \varphi_0'(x,y)$ for arbitrary $\varphi_0 \in M_{(\Gamma,x)}$*

Proof. By the lemma 5, we know that $M_{(\Gamma,x)}$ is not an empty set.
On one hand, for $\forall \varphi_0 \in M_{(\Gamma,x)}$

$$\begin{aligned} \Gamma^{'}(x,y) &= \lim_{\alpha\to 0^+} \frac{\Gamma(x+\alpha y)-\Gamma(x)}{\alpha} \\ &= \lim_{\alpha\to 0^+} \frac{\sup\limits_{\varphi\in H}\varphi(x+\alpha y)-\sup\limits_{\varphi\in H}\varphi(x)}{\alpha} \\ &\geq \lim_{\alpha\to 0^+} \frac{\varphi_0(x+\alpha y)-\varphi_0(x)}{\alpha} \\ &= \varphi_{0'}(x,y) \end{aligned} \tag{5}$$

On the other hand, for any $\alpha > 0$, by the definition of *supremum*, we get that there exists a $\varphi_1 \in H$ such that

$$\Gamma(x+\alpha y) = \sup_{\varphi\in H}\varphi(x+\alpha y) \leq \varphi_1(x+\alpha y)+\alpha^2.$$

Furthermore, by using $\varphi_1(x) \leq \varphi_0(x) = \Gamma(x)$ and the continuity of $\varphi_i(i=0,1)$, there exists an $\bar{\alpha} > 0$ such that $\varphi_1(x+\alpha y) \leq \varphi_0(x+\alpha y)$, $\alpha \in (0,\bar{\alpha})$, and also we have

$$\Gamma(x+\alpha y) \leq \varphi_1(x+\alpha y)+\alpha^2 \leq \varphi_0(x+\alpha y)+\alpha^2.$$

Thus we have

$$\begin{aligned} \Gamma^{'}(x,y) &= \lim_{\alpha\to 0^+} \frac{\Gamma(x+\alpha y)-\Gamma(x)}{\alpha} \\ &\leq \lim_{\alpha\to 0^+} \frac{\varphi_0(x+\alpha y)-\varphi_0(x)+\alpha^2}{\alpha} \\ &= \varphi_0^{'}(x,y) \end{aligned} \tag{6}$$

By the inequalities (5) and (6), we have $\Gamma^{'}(x,y) = \varphi_0^{'}(x,y), \quad \forall \varphi_0 \in M_{(\Gamma,x)}$. □

Assume $\varphi \in H$ is convex and real-valued function, then we have the following lemma.

Lemma 7. $\Gamma = \sup\limits_{\varphi\in H}\varphi$ *is also a convex function on X.*

Proof. Let $\alpha \in [0,1]$, for arbitrary $x_1, x_2 \in X$, we have that

$$\begin{aligned} \Gamma(\alpha x_1+(1-\alpha)x_2) &= \sup_{\varphi\in H}\varphi(\alpha x_1+(1-\alpha)x_2) \\ &\leq \sup_{\varphi\in H}(\alpha\varphi(x_1)+(1-\alpha)\varphi(x_2)) \\ &\leq \alpha\sup_{\varphi\in H}\varphi(x_1)+(1-\alpha)\sup_{\varphi\in H}\varphi(x_2) \\ &= \alpha\Gamma(x_1)+(1-\alpha)\Gamma(x_2) \end{aligned}$$

we get,hence, that Γ is a convex function on X. □

Theorem 8. *Let H be a closed and bounded set with respect to g_0, $\varphi \in H$ be convex and real-valued function, Γ defined as the above, $g_0 \in G$, then the following statements are equivalent,*

(1) *g_0 is a sun-point of Γ in G,*
(2) $g_0 \in P_{(\Gamma,G)} \Leftrightarrow \Gamma^{'}(g_0, g-g_0) \geq 0$,
(3) $g_0 \in P_{(\Gamma,G)} \Leftrightarrow \varphi_0^{'}(g_0, g-g_0) \geq 0$ *for all* $\varphi_0 \in M_{(\Gamma,g_0)}$.
(4) $g_0 \in P_{(\Gamma,G)}$ *and* $\varphi_0 \in M_{(\Gamma,g_0)} \Leftrightarrow (\varphi_0, g_0)$ *is a GSP of* Ψ *in* (H,G).

Proof. We define a function

$$\gamma(\lambda) = \frac{\Gamma(g_0 + \lambda(g-g_0)) - \Gamma(g_0)}{\lambda},$$

for convenience to prove the theorem, where $\lambda \in [0,1]$.

(1)⇒(2)

If $g_0 \in P_{(\Gamma,G)}$, then we have that $g_0 \in P_{(\Gamma,\tilde{G}_{g_0})}$ by condition (1). Hence, $\forall g \in G$, we obtain

$$\Gamma(g_0 + \lambda(g-g_0)) \geq \Gamma(g_0), \qquad \lambda \in [0,1]$$

So for any $\lambda \in (0,1]$, we get $\gamma(\lambda) \geq 0$, which implies $\Gamma^{'}(g_0, g-g_0) \geq 0$.

Conversely, we know that Γ is a convex function on X by Lemma 7. So for $0 < \mu < \nu$, we get

$$\begin{aligned} \Gamma(g_0 + \mu(g-g_0)) &= \Gamma\left(\frac{\mu}{\nu}(g_0 + \nu(g-g_0)) + \frac{\nu-\mu}{\nu} g_0\right) \\ &\leq \frac{\mu}{\nu}\Gamma(g_0 + \nu(g-g_0)) + \frac{\nu-\mu}{\nu}\Gamma(g_0), \end{aligned}$$

which implies $\gamma(\mu) \leq \gamma(\nu)$, that is to say, γ is an increasing function on $[0,1]$. Hence, when $\lambda \to 0$, we have

$$\gamma(\lambda) \geq \Gamma^{'}(g_0, g-g_0) \geq 0.$$

When $\lambda = 1$, we get $\Gamma(g) \geq \Gamma(g_0)$.

(2)⇒(1)

let $g_0 \in P_{(\Gamma,G)}$, from condition (2) we get $\Gamma^{'}(g_0, g-g_0) \geq 0$ for all $g \in G$, which means for any positive number $\lambda \in [0,1]$, $\gamma(\lambda) \geq 0$, namely, $\Gamma(g_0 + \lambda(g-g_0)) \geq \Gamma(g_0)$, that is to say $g_0 \in P_{(\Gamma,\tilde{G}_{g_0})}$.

It is evident to see that the equivalence between the statements (2) and (3) is established by using Lemma 6.

(3)⇒(4)

Assume $g_0 \in P_{(\Gamma,G)}$, then we get $\varphi_0^{'}(g_0, g-g_0) \geq 0$ for all $\varphi_0 \in M_{(\Gamma,g_0)}$ from (3). Let

$$\gamma(\alpha) = \frac{\varphi_0(g_0 + \alpha(g-g_0)) - \varphi_0(g_0)}{\alpha},$$

where $\alpha \in (0,1]$. When φ_0 is a convex function, It is also been able to prove that $\gamma(\alpha)$ is an increasing function on $(0,1]$. So we have $\gamma(1) \geq \lim\limits_{\alpha \to 0^+} \gamma(\alpha) \geq 0$, which implies

$$\varphi_0(g) \geq \varphi_0(g_0). \tag{7}$$

Furthermore, we can get

$$\varphi(g_0) \leq \sup_{\varphi \in H} \varphi(g_0) = \varphi_0(g_0), \tag{8}$$

from $\varphi_0 \in M_{(\Gamma, g_0)}$. By the inequality (7)(8), we have

$$\Psi(\varphi, g_0) \leq \Psi(\varphi_0, g_0) \leq \Psi(\varphi_0, g), \qquad (\varphi, g) \in (H, G).$$

$(4) \Rightarrow (3)$

Assume (φ_0, g_0) is a GSP of Ψ in (H, G), we have

$$\Psi(\varphi, g_0) \leq \Psi(\varphi_0, g_0) \leq \Psi(\varphi_0, g),$$

namely

$$\varphi(g_0) \leq \varphi_0(g_0) \leq \varphi_0(g). \tag{9}$$

Therefor, by inequality (9) and convexity of φ_0, we can get

$$\begin{aligned}
\varphi_0'(g_0, g - g_0) &= \lim_{\alpha \to 0^+} \frac{\varphi_0(g_0 + \alpha(g - g_0)) - \varphi_0(g_0)}{\alpha} \\
&\geq \lim_{\alpha \to 0^+} \frac{(1-\alpha)\varphi_0(g_0) + \alpha\varphi_0(g) - \varphi_0(g_0)}{\alpha} \\
&= \lim_{\alpha \to 0^+} \frac{\alpha[\varphi_0(g) - \varphi_0(g_0)]}{\alpha} \\
&= \varphi_0(g) - \varphi_0(g_0) \geq 0.
\end{aligned}$$

Hence, the equivalence between GSP and sun-point is established when functions are convex. □

4 Future Work

GSP and sun-point are all to depict optimum solution of a optimization problem without using directional derivative from their concepts, but the equivalence between them has been established with the help of directional derivative in this paper. Now we bring forward the following questions, one is whether convexity is necessary in the course of proof; another is whether we can find a way out to established the equivalence without using directional derivative. Our part of future work will be furthered in these prolems.

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