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An Optimal Investment Policy in Equity-Debt Financed Firms with Finite Maturities

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Abstract In this paper we examine an optimal investment policy of the firm, which is financed by issuing equity and debt during a period of time, using real options framework. We examine the effects of the maturity of investment on the values of equity, debt, firm, tax shield and bankruptcy cost. Specifically, we show that the investment timing depends not only on the investment threshold but also on the coupon payment. We also show that the optimal leverage ratio is determined independently of investment decision.

Keywords Investment timing; Finite maturity; Debt financing; Capital structure; Real options

1 Introduction

Real options theory, pioneered by Brennan and Schwartz [1], and McDonald and Siegel [2], and summarized in Dixit and Pindyck [3], has attracted growing attention because it enables us to account for the value of flexibility under uncertainty. In standard real options models, all-equity financing is assumed, and the interactions between corporate financing and investment decisions have been not analyzed.

Recently, a number of studies have investigated the interaction among investment and financing decisions of a firm under uncertainty by means of real options framework. In the literature, investment problems for the firm with growth options, which is financed with equity and debt are investigated (e.g. Lyandres and Zhdanov [4], Mauer and Ott [5], Mauer and Sarkar [6], Sundaresan and Wang [7] and Zhdanov [8]). In these studies, in order to simplify the problem the infinite maturity of the investment and debt are assumed. However, in the real case firms usually have investment policies with finite maturity. For example, in the investment of oil reserves development, offshore leases limit the time before the development. This implies that the option to develop the oil reserves appears to be not a perpetual one [9]. Likewise, in the investment of the power plant, it is difficult to postpone the construction investment for a long term because of ensuring the stable supply. Therefore, when the investment problem as real case is analyzed, it is necessary to consider the maturity of the investment decision. Furthermore, it seems that not only the investment timing but also financing and capital structure are dependent on the maturity.

In this paper we examine the optimal investment policy of the firm that is financed by issuing equity and debt during a fixed period. We show that the maturity depends not only on the investment threshold but also on the coupon payment. Furthermore, we examine the effect of the tax shield and the bankruptcy cost with respect to time. Consequently, we show the optimal leverage ratio is determined, independently of investment decision, from the optimal coupon payment maximizing the firm value.

2 The Model

Consider a firm with an option to invest at any time by paying a fixed investment cost I. The firm partially finances the cost of investment with debt with the instantaneous contractual coupon payment of c and infinite maturity. The coupon payment is tax-deductible at a constant corporate tax rate λ. We suppose that the firm observes the demand shock X_t for its product, where X_t is given by a geometric Brownian motion,

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \tag{1}$$

where μ and σ are the risk-adjusted expected growth rate and the volatility of X_t, respectively, and W_t is a standard Brownian motion defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Once the investment option is exercised, we assume that the firm can receive the instantaneous profit,

$$\pi(X_t) = (1-\lambda)(QX_t - c), \tag{2}$$

where $Q > 0$ is the quantity produced from the asset in place.

In order to examine the interaction of debt issuing and the investment maturity, we consider the cases in which the investments with the infinite and finite maturities are financed with all-equity and with equity and debt.

2.1 Investment Option with Infinite Maturity

2.1.1 All-equity Financing

First, we assume that the investment is financed entirely with equity, i.e. the coupon payment equals zero, $c = 0$. This case has been investigated by using the basic model in real options theory such as in McDonald and Siegel [2] and Dixit and Pindyck [3].

The optimal investment rule is to exercise the investment option at the first passage time of the stochastic shock to an optimal investment threshold x^*. Let $\mathscr{T}_{t_1,t_2}$ be the set of stopping times with respect to the filtration as $\{\mathscr{F}_s; t_1 \leq s \leq t_2\}$, $\tau \in \mathscr{T}_{0,\infty}$ be the investment time (the stopping time). Supposing that the firm can perpetually receive the profit after the investment, the value of an investment option $F(x)$ can be formulated as

$$F(x) = \sup_{\tau \in \mathscr{T}_{0,\infty}} \mathbb{E}_0^x \left[\int_\tau^\infty e^{-ru}(1-\lambda)QX_u du - e^{-r\tau} I \right], \tag{3}$$

where $\mathbb{E}_t^x$ is the conditional expectation operator upon $X_t = x$, and r is the risk-free interest rate. In order to ensure that the firm's value is finite, we assume $r > \mu$. Using standard arguments of real options framework, the value of the investment option $F(x)$ is given by

$$F(x) = \left(\frac{x}{x^*}\right)^{\beta_1} (\varepsilon(x^*) - I), \quad x < x^*, \tag{4}$$

and the investment threshold x^* is given by

$$x^* = \frac{\beta_1}{\beta_1 - 1}\frac{r-\mu}{Q(1-\lambda)}I, \tag{5}$$

where $\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1$ and $\varepsilon(x)$ is the total post-investment profit in which the investment is financed entirely with equity,

$$\varepsilon(x) = \frac{1-\lambda}{r-\mu}Qx.$$

2.1.2 Equity and Debt Financing

Next, following Mauer and Sarkar [6], and Sundaresan and Wang [7], we consider a firm which has an option of the investment with the infinite maturity, which is financed with equity and debt ($c > 0$),

We model the values of equity and debt with coupon payment c after the exercise of investment option. Once the investment option has been exercised, the optimal default policy is established from the issue of debt. The optimal default strategy of the equity holders selects the optimal default timing, maximizing the equity value. Let $E(x;c)$ be the total value of equity issued at time t and $\tau_d \in \mathscr{T}_{0,\infty}$ the default time. The value of equity $E(x;c)$ is equal to zero at the default time τ_d. The optimization problem of the equity holders can be formulated by

$$E(x;c) = \sup_{\tau_d \in \mathscr{T}_{t,\infty}} \mathbb{E}_t^x\left[\int_t^{\tau_d} e^{-r(u-t)}(1-\lambda)(QX_u - c)du\right]. \tag{6}$$

The optimal default time τ_d^* is given by

$$\tau_d^* = \inf\{\tau_d \in [t,\infty) \mid X_{\tau_d} \le x_d\}, \tag{7}$$

where x_d is the optimal default threshold.

Using standard arguments, the equity value $E(x;c)$ is given by

$$E(x;c) = \begin{cases} \varepsilon(x) - \frac{(1-\lambda)c}{r} - \left(\varepsilon(x_d) - \frac{(1-\lambda)c}{r}\right)\left(\frac{x}{x_d}\right)^{\beta_2}, & x > x_d \\ 0, & x \le x_d \end{cases} \tag{8}$$

and the default threshold x_d is

$$x_d = \frac{\beta_2}{\beta_2 - 1}\frac{r-\mu}{Q}\frac{c}{r}, \tag{9}$$

where $\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0$.

Let $D(x;c)$ be the total value of debt issued at investment time t. Since the holders of debt can receive the continuous coupon payment of c, the value of debt is given by

$$D(x;c) = \mathbb{E}_t^x\left[\int_t^{\tau_d^*} e^{-r(u-t)}c\,du + e^{-r(\tau_d^*-t)}(1-\theta)\varepsilon(X_{\tau_d^*})\right], \tag{10}$$

where θ is the proportional bankruptcy cost, $0 \leq \theta \leq 1$. The holders of debt are entitled to the unlevered value of the firm net of proportional bankruptcy cost, $(1-\theta)\varepsilon(x)$. Using the optimal default threshold x_d, the value of debt can be represented by

$$D(x;c) = \frac{c}{r} - \left(\frac{c}{r} - (1-\theta)\varepsilon(x_d)\right)\left(\frac{x}{x_d}\right)^{\beta_2}, \; x > x_d. \tag{11}$$

The sum of $E(x;c)$ in Eq. (8) and $D(x;c)$ in Eq. (11) gives the firm value as

$$\begin{aligned} V(x;c) &= E(x;c) + D(x;c) \\ &= \varepsilon(x) + TS(x;c) - BC(x;c), \;\; x > x_d, \end{aligned} \tag{12}$$

where $TS(x;c)$ and $BC(x;c)$ represent the expected present value of debt tax shields and the expected present value of bankruptcy cost, given by

$$TS(x;c) = \frac{\lambda c}{r}\left(1 - \left(\frac{x}{x_d}\right)^{\beta_2}\right), \tag{13}$$

$$BC(x;c) = \theta\varepsilon(x_d)\left(\frac{x}{x_d}\right)^{\beta_2}. \tag{14}$$

We consider the optimal investment policy maximizing the firm value given in Eq. (12). Let $c^*(x)$ be the optimal coupon payment of debt issued when $X_t = x$ at time t. The optimal capital structure is achieved by selecting the optimal coupon payment. The optimal investment timing and the optimal coupon payment can be determined by maximizing the value of investment option, simultaneously. The value of the investment option $F(x)$ can be formulated as

$$F(x) = \sup_{\tau \in \mathscr{T}_{0,\infty}, c>0} \mathbb{E}_0^x \left[e^{-r\tau}(V(X_\tau;c) - I)\right]. \tag{15}$$

Following Sundaresan and Wang [7], the optimal coupon payment $c^*(x)$ for any x is represented by

$$c^*(x) = \arg\max_{c>0} V(x;c) = \frac{\beta_2 - 1}{\beta_2}\frac{r}{r-\mu}\frac{Qx}{h} > 0, \tag{16}$$

where h is

$$h = \left(1 - \beta_2\left(1 - \theta + \frac{\theta}{\lambda}\right)\right)^{\frac{1}{\beta_2}}.$$

The value of the investment option $F(x)$ is represented by

$$F(x) = \begin{cases} \left(\frac{x}{x^*}\right)^{\beta_1}\{V(x^*;c^*(x^*)) - I\}, & x < x^*, \\ V(x;c^*(x)) - I, & x \geq x^* \end{cases} \tag{17}$$

and the optimal investment threshold x^* is given by

$$x^* = \frac{\beta_1}{\beta_1 - 1}\frac{r-\mu}{Q}\frac{\psi}{1-\lambda}I, \tag{18}$$

where ψ is

$$\psi = \left(1 + \frac{\lambda}{(1-\lambda)h}\right)^{-1} < 1.$$

2.2 Investment Option with Finite Maturity

In this section, we consider the firm with an option for investment with the finite maturity $T < \infty$.

2.2.1 All-equity Financing

First, we assume the all-equity financing. Let $\tau \in \mathscr{T}_{0,T}$ be the investment time (the stopping time). The value of the investment option at time $t \in [0,T]$ is formulated by

$$F(x,t) = \sup_{\tau \in \mathscr{T}_{t,T}} \mathbb{E}_t^x \left[e^{-r(\tau-t)} \left(\int_\tau^\infty e^{-r(u-\tau)}(1-\lambda)QX_u du - I \right)^+ \right], \tag{19}$$

where $(x)^+ = \max(x,0)$. Denote x_t^* as the optimal investment threshold at time t. From Bellman equation, the value of the investment option with finite maturity T satisfies the partial differential equation

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} + \mu x \frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} - rF = 0, \;\; x < x_t^* \tag{20}$$

and the boundary conditions

$$\begin{cases} F(X_T,T) = (\varepsilon(X_T) - I)^+, \\ \lim\limits_{x \uparrow x_t^*} F(x,t) = (\varepsilon(x_t^*) - I)^+, \quad t \in [0,T), \\ \lim\limits_{x \uparrow x_t^*} \dfrac{\partial F}{\partial x}(x,t) = \dfrac{1-\lambda}{r-\mu} Q, \qquad t \in [0,T). \end{cases} \tag{21}$$

The first condition is the terminal condition that ensures the investment option of the firm at the maturity. The second condition is the value-matching condition requiring that the value of investment option at the investment threshold is equal to the post-investment profit minus the investment cost. The last condition is the smooth-pasting condition that ensures the optimality of the the investment threshold x_t^*.

2.2.2 Equity and Debt Financing

Next, we consider a firm which has an option of the investment with the finite maturity $T < \infty$, which is financed with equity and debt. The value of the investment option $F(x,t)$ at time t is given by

$$F(x,t) = \sup_{\tau \in \mathscr{T}_{t,T}, c>0} \mathbb{E}_t^x \left[e^{-r(\tau-t)} \left(V(X_\tau;c) - I \right)^+ \right]. \tag{22}$$

Using the optimal coupon payment given in Eq. (16), the value of the investment option satisfies the partial differential equation as in all-equity financing case and the boundary conditions

$$\begin{cases} F(X_T,T) = (V(X_T;c^*(X_T)) - I)^+, \\ \lim\limits_{x \uparrow x_t^*} F(x,t) = (V(x_t^*;c^*(x_t^*)) - I)^+, \quad t \in [0,T), \\ \lim\limits_{x \uparrow x_t^*} \dfrac{\partial F}{\partial x}(x,t) = \dfrac{dV}{dx}(x_t^*;c^*(x_t^*)), \qquad t \in [0,T). \end{cases} \tag{23}$$

3 Numerical Analysis

In this section, calculation results of the value of equity, debt, firm, the optimal investment threshold, the effects of the tax shield and the bankruptcy cost, the optimal coupon payment and optimal leverage ratio are presented in order to examine the effect of the finite maturity for investment. We use the following base case parameters: $Q = 1$, $x = 0.3$, $\mu = 0.01$, $\sigma = 0.2$, $r = 0.05$, $I = 5$, $c = 0.3$, $\theta = 0.3$, and $\lambda = 0.3$.

Fig. 1 shows the values of equity and debt, $E(x;c)$ and $D_s(x;c)$, as a function of x in $c = 0.3$. The default threshold value, x_d, is 0.139. These values can provide the investment value of the firm financed by issuing equity and debt. In Fig. 2, the values of investment option in $c = 0.3$ are shown for the maturities $T = 1$, 5, 10 and ∞, as a function of x at initial time $t = 0$ at initial time 0 in the equity-debt financing case of Secs. 2.1.2 and 2.2.2. As can be seen in this figure, the investment value increases as the maturity is longer.

Fig. 3 shows the optimal investment threshold value, x_t^*, for several coupon payments $c = 0.0$, 0.1, 0.3 and 0.5 as a function of time t when the maturity, T, is 50. As shown in this figure, the degree of the threshold for coupon payments appears to be dependent on time. This implies that the investment timing depends not only on the investment threshold but also on the coupon payment. Hence, it turns out that it is important to consider the maturity of investment in the investment policy of the firm financed by equity and debt.

In Figs. 4 and 5, the expected present value of tax shield $TS(x;c)$ in Eq. (13) and that of bankruptcy cost $BC(x;c)$ in Eq. (14), and the ratios of them to the firm value; $TS(x;c)/V(x;c)$ and $BC(x;c)/V(x;c)$ at the investment threshold x_t^* are shown as a function of time t. We consider the cases in which the coupon payments are $c = 0.5$ and $c = 0.1$. The value of tax shield decreases more slowly than the firm value with respect to time. Hence, although the value of tax shield decreases, the ratio of them to the firm value increases with time. On the other hand, both the value and the rate of bankruptcy cost increase in time. Since the threshold value of the investment with the short maturity is small, the possibility of bankruptcy is high, and consequently, the bankruptcy cost is also high.

Fig. 6 shows the debt value $D(x_t^*;c^*(x_t^*))$, the firm value $V(x_t^*;c^*(x_t^*))$, the optimal coupon payment $c^*(x_t^*)$ and the optimal leverage ratio $D(x_t^*;c^*(x_t^*))/V(x_t^*;c^*(x_t^*))$ on the optimal investment threshold and optimal coupon payment with respect to time t. Although the debt value, the firm value and the optimal coupon payment decrease with time, the optimal leverage ratio is constant. Since the optimal coupon payment maximizing the firm value adjusts the capital structure, the optimal leverage ratio is determined independently of investment decision. If the debt has the finite maturity, the optimal leverage will depend on time. We require consideration of both the maturity of the investment and that of the debt.

4 Concluding Remarks

In this paper, we have investigated the optimal investment policy of the firm that is financed by issuing equity and debt during a fixed period. The value of investment for the firm with finite maturity increases and the investment timing is later as the remaining time to maturity becomes long. Although the value of tax shields decreases with respect to time, the value of bankruptcy cost and the ratio of them to the firm value increase. We

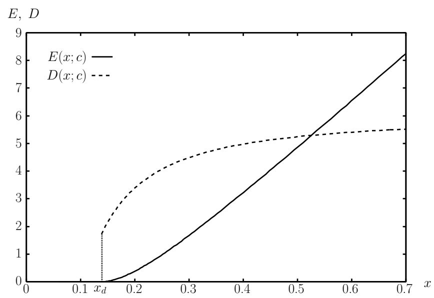

Figure 1: Equity and debt values ($c = 0.3$)

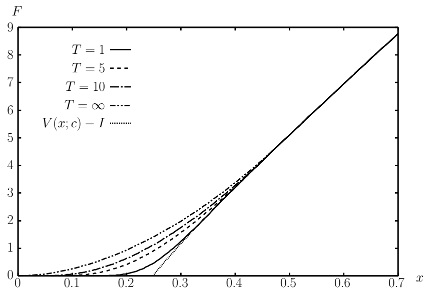

Figure 2: The values of investment option for the firm issuing equity and debt for each maturity ($c = 0.3$)

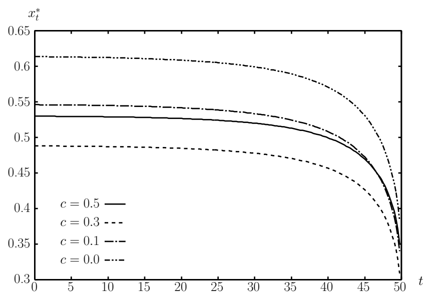

Figure 3: Optimal investment thresholds for each coupon payment

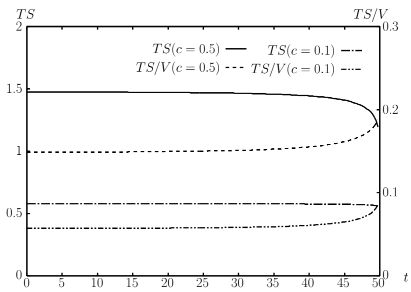

Figure 4: The value and the ratio of tax shield

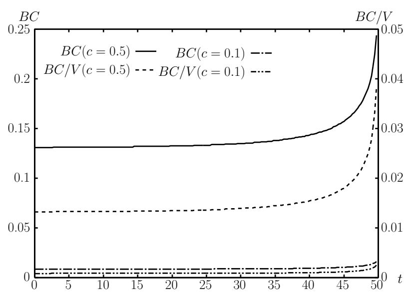

Figure 5: The value and the ratio of bankruptcy cost

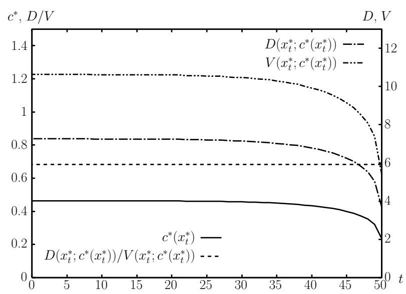

Figure 6: The optimal coupon payment, leverage, debt value and firm value

show that the investment timing depends on the coupon payment, and that the optimal leverage ratio is determined independently of investment decision. Since many debt contracts include the finite maturity, in the future, we will examine the effect for both the investment maturity and the debt maturity. In addition, possible extension of this study also includes the term structure of interest rate such as CIR model and Vasicek model.

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