The Ninth International Symposium on Operations Research and Its Applications (ISORA'10)
Chengdu-Jiuzhaigou, China, August 19–23, 2010

Equitable Δ-Coloring of Planar Graphs without 4-cycles

Xiang Tan*

School of Statistics and Mathematics, Shandong University of Finance,
Jinan, Shandong, 250014, China

Abstract In this paper, we prove that if G is a planar graph with maximum degree $\Delta \geq 7$ and without 4-cycles, then G is equitably m-colorable for any $m \geq \Delta$.

1 Introduction

In this paper, all graphs are finite, simple and undirected. Any undefined notation follows that of Bondy and Murty [1]. Let G be a graph, we use $V(G)$, $|G|$, $E(G)$, $e(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, order, the edge set, size, the maximum (vertex) degree and the minimum (vertex) degree of G, respectively. For subsets U and W of $V(G)$, we denote by $e(U,W)$ the number of edges with one ends in U and the other in W. If $U = \{v\}$, we write $e(\{v\},W)$ for $e(v,W)$. A subset V' is called an independent s-set of G if $|V'| = s$ and no two vertices of V' are adjacent in G. Let $G \cup H$ denote the union of two vertex-disjoint graphs G and H. For a planar graph G, the degree of a face f, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. And we use Φ and r_i to denote the number of faces and i-faces in the planar graph G, respectively.

An *equitable k-coloring* of a graph G is a proper k-coloring, for which any two color classes differ in size by at most one. If f is an equitable coloring of G using k colors, then we say that f is an equitable k-coloring of G. The least integer k for which G has an equitable k-coloring is defined to be the *equitable chromatic number* of G and denoted by $\chi_e(G)$. The least integer k for which G has an equitable k'-coloring of G for every $k' \geq k$ is denoted by $\chi^*(G)$.

Hajnal and Szemerédi [5] proved that any graph with maximum degree $\Delta(G) \leq m$ has an equitable $(m+1)$-coloring. In 1994, Chen, Lih and Wu [3] proved that G is equitably Δ-colorable if G is a connected graph with $\Delta(G) \geq \frac{|G|}{2}$ or $\Delta(G) \leq 3$ and G is different from K_m and $K_{2m+1,2m+1}$ for all $m \geq 1$. And based on the above result, they put forth the following conjecture:

Conjecture 1. A connected graph G is equitably $\Delta(G)$-colorable if it is different from K_m, C_{2m+1}, and the complete bipartite graph $K_{2m+1,2m+1}$ for all $m \geq 1$.

*Corresponding author. E-mail address: xtandw@126.com

For planar graphs, Yap and Zhang [9] proved that a planar graph is equitably m-colorable for any $m \geq \Delta(G) \geq 13$, and they also proved in [8] that Conjecture 1 is true for outerplanar graphs. In this paper, we prove the following theorem:

Theorem 1. Let G be a planar graph with $\Delta(G) \geq 7$ and without 4-cycles. Then G is equitably m-colorable for any $m \geq \Delta(G)$.

2 Some useful Lemmas

Let us introduce some notations and definitions, they are similar to that of [6]. Let G be a graph with mt vertices. A *nearly equitable* m-coloring of G is a proper coloring f, whose color classes all have size t except for one small class $V^- = V^-(f)$ with size $t-1$ and one large class $V^+ = V^+(f)$ with size $t+1$. Given such a coloring f, define an auxiliary digraph $\mathscr{H} = \mathscr{H}(f)$ as follows. The vertices of $\mathscr{H}$ are the color classes of f. A directed edge UW belongs to $E(\mathscr{H})$ iff some vertex $y \in U$ has no neighbors in W. In this case we say that y is *movable* to W. Call $W \in V(\mathscr{H})$ *accessible*, if V^- is reachable from W in $\mathscr{H}$. So V^- is trivially accessible. Let $\mathscr{A} = \mathscr{A}(f)$ denote the family of accessible classes, $A := \bigcup \mathscr{A}$, $r := |\mathscr{A}|$ and $B := V(G) \backslash A$.

Lemma 1. If G has a nearly equitable coloring, whose large class V^+ is accessible, then it has an equitable coloring with the same number of colors.

Suppose $V^+ \subseteq B$. Then $|A| = rt-1$, $|B| = (m-r)t+1$ and $d_A(y) \geq r$ for each vertex $y \in B$. For an accessible class $U \in \mathscr{A}$, define $\mathscr{S}_U(f)$ to be the set of classes $X \in \mathscr{A}$ such that there is an XV^--path in $\mathscr{H} - U$ and $\mathscr{T}_U := \mathscr{T}_U(f) := \mathscr{A} \backslash (\mathscr{S}_U(f) + U)$. Call U *terminal*, if $\mathscr{S}_U(f) = \mathscr{A} - U$, otherwise U is $non-terminal$. Trivially, V^- is non-terminal. Choose a non-terminal U such that $|\mathscr{T}_U|$ is minimum and set $\mathscr{A}' := \mathscr{T}_U$. Let $A' := \bigcup \mathscr{A}'(f) := \bigcup \mathscr{A}'$.

Lemma 2. Every class in $\mathscr{A}'$ is terminal.

The proof of Lemma 1 and Lemma 2 can be found in [6].

An edge zy is *solo* if $z \in W \in \mathscr{A}'$, $y \in B$ and $N_W(y) = \{z\}$. The ends of solo edges are called *solo vertices* and vertices linked by solo edges are called *special neighbors* of each other. Let S_z denote the set of special neighbors of z.

Lemma 3. Let G be a planar graph of order mt and without 4-cycles. Then $e(G) \leq \frac{15}{7}mt - \frac{30}{7}$ and $\delta(G) \leq 4$.

Proof. We need only to consider the case that G is connected. Since G contains no 4-cycles, it doesn't contain adjacent 3-cycles, we have $3r_3 \leq e(G)$. Thus $5\Phi - 2r_3 = 5(r_3 + r_5 + \cdots + r_n) - 2r_3 \leq 3r_3 + 5r_5 + \cdots + nr_n = \sum_{f \in F} d(f) = 2e(G)$. We have $\Phi \leq \frac{8e(G)}{15}$. By Euler's formula $|G| - e(G) + \Phi = 2$, we have $e(G) \leq \frac{15}{7}(|G|-2) = \frac{15}{7}mt - \frac{30}{7}$. In

[2], Borodin proved that $\delta(G) \leq 4$ for each plane graph without adjacent triangles. It completes the proof of Lemma 3.

Lemma 4. Every planar graph without 4-cycles is 4-choosable.

The proof of Lemma 4 can be found in [7].

Lemma 5. Let m and s be positive integers. Suppose G is a planar graph with $\Delta(G) \leq m$ and without 4-cycles. If G has an independent s-set V' and there exists $C \subseteq V(G) \backslash V'$ such that $|C| > \frac{s(m+2)}{2}$ and $e(v, V') \geq 1$ for any $v \in C$, then C contains two nonadjacent vertices α and β which are adjacent to exactly one and the same vertex $\gamma \in V'$.

Proof. Let $C_1 \subseteq C$ be such that each $v \in C_1$ is adjacent to exactly one vertex of V'. Let $|C_1| = r$. Then $r + 2(|C| - r) \leq e(C, V') \leq ms$, from which it follows that $r \geq 2|C| - ms > 2s$. Hence V' contains at least one vertex γ which is adjacent to at least three vertices of C_1. As G is a planar graph without 4-cycles, C_1 contains two nonadjacent vertices α and β which are adjacent to γ.

Lemma 6. Let G be a planar graph with maximum degree $\Delta(G) \leq m$ and without 4-cycles, $|G| = mt$. Let f be a nearly equitable coloring of G and $V^+(f) \subseteq B$. If $|B| = (m-r)t + 1 > \frac{mt}{2}$ and $r \geq 3$, then there exists a solo vertex $z \in W \in \mathscr{A}'$ such that either z is movable to a class in $\mathscr{A} \backslash \{W\}$ or z has two nonadjacent neighbors in B.

Proof. Suppose not. Let S be the set of solo vertices in W. As G is a planar graph without 4-cycles, we can get $|S_z| \leq 2$ for any $z \in S$. Then there exists at most $2|S|$ vertices in B which has exactly one neighbor in W, thus $e(W, B) \geq 2|S| + 2(|B| - 2|S|) = 2|B| - 2|S|$. And since no vertex in S is movable to a class in $\mathscr{A} \backslash \{W\}$, each $z \in S$ satisfies $d_A(z) \geq r - 1$, we can get $d_B(z) \leq m - (r-1)$. Thus $2|B| - 2|S| \leq e(W, B) \leq [m - (r-1)]|S| + m(t - |S|)$. It follows that $2|B| - mt + (r-3)|S| \leq 0$, a contradiction.

Lemma 7. Let m and t be positive integers. Let H be a graph of order mt with vertex chromatic number $\chi \leq m$. If $e(H) \leq (m-1)t$, then H is equitably m-colorable.

The Proof of Lemma 7 can be found in [9].

Lemma 8. Let $m \geq 5$ and $t \geq 2$ be integers. Let G be a planar graph with maximum degree $\Delta(G) \leq m$ and without 4-cycles, $|G| = mt$. If $e(G) \leq (\frac{1}{7}m + \frac{30}{7})t + m - 3$, then G is equitably m-colorable.

Proof. Suppose for a contradiction, that G is an edge-minimal counterexample to the lemma. As G is planar and without 4-cycles, by Lemma 3, G has an edge $xy \in E(G)$ where $d(x) \leq 4$. By minimality, $G - xy$ has an equitable m-coloring ϕ which has color classes $V_1, V_2, \cdots, V_m$, where $|V_i| = t$ for $i = 1, 2, \cdots, m$. Clearly we only need to consider the case that x, y are in the same color class. Without loss of generality, assume $x, y \in V_1$ and $N(x) \subseteq V_1 \cup V_2 \cup \cdots \cup V_4$. Let $V^- = V_1 \backslash \{x\}$, $V^+ = V_m \cup \{x\}$. Thus we get a nearly equitable m-coloring f of G. Clearly $V^+ \subseteq B$ by Lemma 1. If there exists V_j for some

$5 \le j < m$, such that V_j is accessible, let $\mathscr{P}$ be a path in $\mathscr{H}$ from V_j to V^-. Since V^+ contains a vertex x that has no neighbors in V_j, $\mathscr{P}_1 = \mathscr{P} \cup \{V^+V_j\}$ is a path from V^+ to V^- in $\mathscr{H}$. Thus G has an equitable m-coloring by Lemma 1, a contradiction. So we can get $r = |\mathscr{A}(f)| \le 4$, $\bigcup_{j=5}^{m} V_j \cup \{x\} \subseteq B(f)$.

Case 1. $r = 4$. Then $|A| = 4t-1$, $|B| = (m-4)t+1$ and $e(A,B) \ge 4[(m-4)t+1]$. Let $A^+ = A \cup \{x\}$.

If $e(G[A]) \le 3t-4$, then $e(G[A^+]) \le 3t$. By Lemma 7, $G[A^+]$ is equitably 4-colorable. Consequently G is equitably m-colorable.

Otherwise, $e(G[A]) > 3t-4$. Then $e(G) \ge e(A,B) + e(G[A]) > 4[(m-4)t+1] + (3t-4) = (4m-13)t \ge e(G)$, a contradiction.

Case 2. $r = 3$ or $r = 2$. In this case, $e(A,B) \ge \min\{3(m-3)t+3, 2(m-2)t+2\} > e(G)$, a contradiction.

Case 3. $r = 1$. Then $e(A,B) \ge (m-1)t+1$ and $|B| = (m-1)t+1 > \frac{(t-1)(m+2)}{2}$. By Lemma 5, there exist two nonadjacent vertices α and β in B which are adjacent to exactly one and the same vertex $\gamma \in V^-$. Let $G_1 = G[B \backslash \{\alpha,\beta\} \cup \{\gamma\}]$ and $G_2 = G[V^- \backslash \{\gamma\} \cup \{\alpha,\beta\}]$. Then $|G_1| = (m-1)t$, $\Delta(G_1) \le m-1$ and $e(G_1) \le e(G[B]) + m - 2 \le (\frac{1}{7}m + \frac{30}{7})t + m - 3 - [(m-1)t+1] + m - 2 = (\frac{37}{7} - \frac{6}{7}m)t + 2m - 6 \le (m-2)t$. Thus G_1 is equitably $(m-1)$-colorable by Lemma 7. Consequently G is equitably m-colorable.

Lemma 9. Let $m \ge 6$ and $t \ge m-4$ be integers. Let G be a planar graph with maximum degree $\Delta(G) \le m$ and without 4-cycles, $|G| = mt$. If $e(G) \le (\frac{8}{7}m + \frac{15}{7})t + m - \frac{44}{7}$, then G is equitably m-colorable.

Proof. Suppose for a contradiction, that G is an edge-minimal counterexample to the lemma. Similar to the proof of Lemma 8, we can get a nearly equitable m-coloring f of G such that $r = |\mathscr{A}(f)| \le 4$, $\bigcup_{j=5}^{m} V_j \cup \{x\} \subseteq B(f)$.

Case 1. $r = 4$. Then $|A| = 4t-1$, $|B| = (m-4)t+1$ and $e(A,B) \ge 4[(m-4)t+1]$. Let $A^+ = A \cup \{x\}$.

If $e(G[A]) \le 3t-4$, then $e(G[A^+]) \le 3t$. By Lemma 7, $G[A^+]$ is equitably 4-colorable, Consequently G is equitably m-colorable.

Otherwise, $e(G[A]) > 3t-4$. Then $e(G) > 4[(m-4)t+1] + (3t-4) = (4m-13)t > e(G)$, a contradiction.

Case 2. $r = 3$. In this case, $e(A,B) \ge 3(m-3)t+3 > e(G)$, a contradiction.

Case 3. $r = 2$. Then $e(A,B) \ge 2(m-2)t+2$ and $|B| = (m-2)t+1 > \frac{(t-1)(m+2)}{2}$. By Lemma 5, there exist two nonadjacent vertices α and β in B which are adjacent to exactly one and the same vertex $\gamma \in V^-$. Let $G_1 = G[B \backslash \{\alpha,\beta\} \cup \{\gamma\}]$ and $G_2 = G[(V^- \backslash \{\gamma\} \cup \{\alpha,\beta\}) \cup (A \backslash \{V^-\})]$. Then $|G_1| = (m-2)t$, $\Delta(G_1) \le m-2$ and $e(G_1) \le e(G[B]) + m - 2 \le (\frac{8}{7}m + \frac{15}{7})t + m - \frac{44}{7} - 2[(m-2)t+1] + m - 2 = (\frac{43}{7} - \frac{6}{7}m)t + 2m - \frac{72}{7} \le (m-3)t$. Thus G_1 is equitably $(m-2)$-colorable by Lemma 7. Consequently G is equitably m-colorable.

Case 4. $r = 1$. Then $e(A,B) \ge (m-1)t+1$ and $|B| = (m-1)t+1 > \frac{(t-1)(m+2)}{2}$. By Lemma 5, there exist two nonadjacent vertices α and β in B which are adjacent to exactly one and the same vertex $\gamma \in V^-$. Let $G_1 = G[B \backslash \{\alpha,\beta\} \cup \{\gamma\}]$ and $G_2 = G[V^- \backslash \{\gamma\} \cup \{\alpha,\beta\}]$. Then $|G_1| = (m-1)t$, $\Delta(G_1) \le m-1$ and $e(G_1) \le e(G[B]) + m - 2 \le (\frac{8}{7}m +$

$\frac{15}{7})t+m-\frac{44}{7}-[(m-1)t+1]+m-2=(\frac{1}{7}m+\frac{22}{7})t+2m-\frac{65}{7}\leq[\frac{1}{7}(m-1)+\frac{30}{7}]t+(m-1)-3$. Thus G_1 is equitably $(m-1)$-colorable by Lemma 8. Consequently G is equitably m-colorable.

3 Proof of Theorem 1

Theorem 1. Let G be a planar graph with maximum degree $\Delta(G)\geq 7$ and without 4-cycles. Then G is equitably m-colorable for any $m\geq\Delta(G)$.

Proof. First we consider the case that $|G|$ is divisible by m. Without loss of generality, assume $|G|=mt$.

Suppose for a contradiction, that G is an edge-minimal counterexample to the Theorem. Similar to that of Lemma 8, we can get a nearly equitable m-coloring f of G, $r=|\mathscr{A}|\leq 4$, $\bigcup_{j=5}^{m}V_j\cup\{x\}\subseteq B$.

Case 1. $r=4$. In this case, $|B|=(m-4)t+1$ and $e(A,B)\geq 4[(m-4)t+1]$. Let $A^+=A\cup x$.

If $e(G[A])\leq 3t-4$, then $e(G[A^+])\leq 3t$. By Lemma 7, $G[A^+]$ is equitably 4-colorable, Consequently G is equitably m-colorable.

Otherwise, $e(G[A])>3t-4$. Then $e(G)>4[(m-4)t+1]+(3t-4)=(4m-13)t>e(G)$, a contradiction.

Case 2. $r=3$. In this case, $e(A,B)\geq 3(m-3)t+3$ and $|B|=(m-3)t+1>\frac{mt}{2}$.

By Lemma 6, there exists a solo vertex $z\in W\in\mathscr{A}'$ and a vertex $y_1\in S_z$ such that either z is movable to a class $X\in\mathscr{A}\backslash\{W\}$, or z is not movable in $\mathscr{A}$ and there exists another vertex $y_2\in S_z$, which is not adjacent to y_1.

Subcase 2.1. z is movable to a class $X\in\mathscr{A}\backslash\{W\}$. Let $G_1=G[A\cup\{y_1\}]$, $G_2=G[B\backslash\{y_1\}]$. Since $W\in\mathscr{A}'(f)$, there exists a path $\mathscr{P}$ from X to $V^-(f)$ in $\mathscr{H}(f)-W$ by Lemma 2. Move z to X and y_1 to $W\backslash\{z\}$ to obtain a nearly equitable 3-coloring φ of G_1 with $V^+(\varphi)=X\cup\{z\}$. Let $\mathscr{P}^*=\mathscr{P}+V^+(\varphi)-X$. Then $\mathscr{P}^*$ is a path from $V^+(\varphi)$ to $V^-(\varphi)$ in $\mathscr{H}(\varphi)$. Thus G_1 has an equitable 3-coloring φ' by Lemma 1. Moreover, $|G_2|=(m-3)t$ and $e(G_2)\leq e(G[B])\leq e(G)-e(A,B)\leq\frac{15}{7}mt-\frac{30}{7}-3[(m-3)t+1]=(9-\frac{6}{7}m)t-\frac{51}{7}\leq(m-4)t$. By Lemma 7, G_2 has an equitable $(m-4)$-coloring g. Then $\varphi'\cup g$ is an equitable m-coloring of G.

Subcase 2.2. z is not movable to any class in $\mathscr{A}$. Then $d_A(z)\geq 2$. Since $W\in\mathscr{A}(f)$, there exists a path $\mathscr{P}$ from W to $V^-(f)$ in $\mathscr{H}(f)$. Let $G_1=G[A\backslash\{z\}\cup\{y_1,y_2\}]$, $G_2=G[B\backslash\{y_1,y_2\}\cup\{z\}]$. Move y_1 and y_2 to $W\backslash\{z\}$ to obtain a nearly equitable 3-coloring φ of G_1 with $V^+(\varphi)=W\backslash\{z\}\cup\{y_1,y_2\}$. Let $\mathscr{P}^*=\mathscr{P}+V^+(\varphi)-W$. As z is not movable to any class in $\mathscr{A}(f)$, we can get $\mathscr{P}^*$ is a path from $V^+(\varphi)$ to $V^-(\varphi)$ in $\mathscr{H}(\varphi)$. Thus G_1 has an equitable 3-coloring φ' by Lemma 1. Moreover, $|G_2|=(m-3)t$ and $e(G_2)\leq e(G[B])+(m-4)\leq\frac{15}{7}mt-\frac{30}{7}-3[(m-3)t+1]+(m-4)=(9-\frac{6}{7}m)t+m-\frac{79}{7}\leq(m-5)t$. Then G_2 has an equitable $(m-4)$-coloring g by Lemma 7. Thus $\varphi'\cup g$ is an equitable m-coloring of G.

Case 3. $r=2$. In this case, $(m-2)t+1\leq e(V^-,B)\leq m(t-1)$, and it follows that $t\geq 4$. Clearly $|B|=(m-2)t+1>\frac{(t-1)(m+2)}{2}$ and $e(v,V^-)\geq 1$ for any $v\in B$. By Lemma 5, there exist two nonadjacent vertices α and β in B which are adjacent to exactly one and the same vertex $\gamma \in V^-$. Let $G_1 = G[B\backslash\{\alpha,\beta\}\cup\{\gamma\}]$ and $G_2 = G[(V^-\backslash\{\gamma\}\cup\{\alpha,\beta\})\cup(A\backslash\{V^-\})]$. Then $|G_1| = (m-2)t$, $\Delta(G_1) \le m-2$ and $e(G_1) \le e(G[B]) + m - 2 \le \frac{15}{7}mt - \frac{30}{7} - 2[(m-2)t+1] + m - 2 = (\frac{1}{7}m+4)t + m - \frac{58}{7} = [\frac{1}{7}(m-2) + \frac{30}{7}]t + (m-2) - \frac{44}{7}$. By Lemma 8, G_1 is equitably $(m-2)$-colorable. Consequently G is equitably m-colorable.

Case 4. $r = 1$. In this case, $(m-1)t + 1 \le e(V^-, B) \le m(t-1)$, and it follows that $t \ge m+1$. Clearly $|B| = (m-1)t + 1 > \frac{(t-1)(m+2)}{2}$. By Lemma 5, there exist two nonadjacent vertices α and β in B which are adjacent to exactly one and the same vertex $\gamma \in V^-$. Let $G_1 = G[B\backslash\{\alpha,\beta\}\cup\{\gamma\}]$ and $G_2 = G[V^-\backslash\{\gamma\}\cup\{\alpha,\beta\}]$. Then $|G_1| = (m-1)t$, $\Delta(G_1) \le m-1$ and $e(G_1) \le e(G[B]) + m - 2 \le \frac{15}{7}mt - \frac{30}{7} - [(m-1)t+1] + m - 2 = (\frac{8}{7}m+1)t + m - \frac{51}{7} = [\frac{8}{7}(m-1) + \frac{15}{7}]t + (m-1) - \frac{44}{7}$. Thus G_1 is equitably $(m-1)$-colorable by Lemma 9. Consequently G is equitably m-colorable.

If $|G|$ is not divisible by m, without loss of generality, assume that $|G| = m(t+1) - j$, where $0 < j < m$. Use induction on $|G|$. As G is planar and without 4-cycles, G has an edge $xy \in E(G)$ where $d(x) \le 4$. By the induction hypothesis, $G - x$ has an equitable m-coloring Φ with color classes $V_1, V_2, \cdots, V_m$, where $|V_i| = t$ or $|V_i| = t+1$. Assume $N(x) \subseteq V_1 \cup V_2 \cup V_3 \cup V_4$. If there exists some $i \ge 5$ such that $|V_i| = t$, then by adding x to V_i to obtain an equitable m-coloring. Otherwise, $|V_i| = t+1$ for any $i \ge 5$, we have $|G| = m(t+1) - j$, $0 < j < 4$. Let $G' = G \cup K_j$, then G' is equitably m-colorable by the above proof, and so is G.

Hence we complete the proof of Theorem 1. □

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