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# Analysis of Two-server Queues with a Variant Vacation Policy

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**Abstract** We study an M/M/2 queueing system with two heterogeneous servers under a variant vacation policy, where the two servers may take together at most $J$ vacations when the system is empty. A quasi-birth-and-death process is formulated to analyze the system. We obtain the explicit expressions of the stationary distribution of the system size and the mean system size. We also show that the number of vacations that the servers take continuously under the condition that the servers are in vacation follows a truncated geometric distribution. The conditional stochastic decomposition properties of the queue length and the waiting time are established for such a system.

**Keywords** Queueing systems; heterogeneous servers; variant vacation policy; matrix-geometric solution; stochastic decomposition.

## 1 Introduction

Queueing systems with vacations have been developed for a wide range of applications in flexible manufacturing systems, service systems and telecommunication systems over more than two decades. Server vacations may occur due to a lack of work, server failure, or another task being assigned to the server. The amount of literature relating to queueing models with vacations is growing rapidly, as can be seen in survey papers by Doshi [1], [2] and Takagi [3].

Most of the literature on multi-server queueing systems generally assume the servers to be homogeneous. This assumption may be valid only when the service process is highly mechanically or electronically controlled. In a queueing system with human servers, the above assumption can hardly be realized. However, there are only a few studies on multi-server queues with vacations in which the service rates of the servers are not identical.

Neuts and Takahashi [4] have pointed out that for the queueing systems with more than two heterogeneous servers, analytical results are intractable. Based on this observation, several authors focused their studies on queues with two heterogeneous servers. Singh [5] studied an M/M/2 queueing system with balking and two heterogeneous servers. Kumar and Madheswari [6] studied an M/M/2 queueing system with two heterogeneous servers and multiple vacations by using the matrix-geometric solution method. Yue *et al.* [7] further considered the model in [6]. They obtained the explicit expression of the rate matrix and presented the conditional stochastic decomposition results for the queue length and the waiting time.

Madan *et al.* [8] studied an M/M/2 queue with Bernoulli schedules and a single vacation policy where the two servers provide heterogeneous exponential service to customers. They obtained the steady-state probability generating functions of the system size for various states of the servers. Yue and Yue [9] studied an M/M/2 queueing system with balking and two heterogeneous servers under Bernoulli schedules and a single vacation policy. They presented a generalization of Model B in Madan *et al.* [8] and obtained the explicit expressions of the steady state condition, the stationary distribution of the system size, and the mean system size.

In this paper, we consider an M/M/2 queueing system with two heterogeneous servers under a variant vacation policy, where the two servers may take together at most $J$ vacations when the system is empty. This type of vacation policy is called a variant vacation policy in [10].

The rest of the paper is organized as follows. Section 2 presents a model description and a quasi-birth-and-death (QBD) model formulation. In Section 3, we obtain the stationary distribution of the system size and the mean system size. We also show that the number of vacations that the servers take continuously under the condition that the servers are in vacation follows a truncated geometric distribution. Section 4 presents the two conditional stochastic decomposition properties for the stationary distribution of the queue length and the waiting time. Conclusions are given in Section 5.

# 2 Model Formulation

In this paper, we consider an M/M/2 queueing system with two heterogeneous servers under a variant vacation policy. The assumptions of the system model are as follows:

Arrivals of customers follow a Poisson process with rate $\lambda$. Arriving customers form a single waiting line based on the order of their arrivals. The total number of potential customers and the system capacity are assumed to be infinite.

Whenever the system becomes empty, the two servers leave for a vacation together with random length $V$. If no customers are found in the queue when the two servers return from the vacation, they again leave for another vacation with the same length. This pattern continues until they return from a vacation to find at least one customer waiting in the queue, or they have already taken $J$ vacations. If no customers are found by the end of the $J$th vacation, the servers stay in the system until one customer arrives. If at least one customer presents in the waiting line for service when the servers return from a vacation, or the servers stay dormant in the system, they immediately start serving the waiting customers until the system becomes empty. The vacation time $V$ follows an exponential distribution with rate $\theta$.

The two servers provide heterogeneous exponential service to customers on a first-come first-serviced (FCFS) basis with service rates $\mu_k$, for the $k$th server, $k = 1,2$. When both servers are free, then a new arriving customer chooses the first server to get the service.

Furthermore, various stochastic processes involved in the system are independent of each other.

Let $N(t)$ be the number of customers in the system at time $t$ and let $L(t) = j$, while $j = 0,1,...,J,J+1$, denotes the status of the servers at time $t$. That is, the state $(i, j)$ means that $i$ ($i \geq 0$) customers are in the system while both servers are taking $(j+1)$th vacation,

$j = 0, 1, .., J-1$. The state $(0, J)$ means that the system is empty while both servers are free. The state $(1, J)$ means that one customer is in the system while Server 1 is busy and Server 2 is free. The state $(1, J+1)$ means that one customer is in the system while Server 1 is free and Server 2 is busy. The state $(i, J)$ means that $i$ $(i \geq 2)$ customers are in the system while both servers are busy. Then $\{(N(t), L(t)), t \geq 0\}$ is a quasi-birth-and-death (QBD) process with a state space denoted by $\Omega$ as follows:

$$\Omega = \{(1, J+1)\} \cup \{(i, j), i \geq 0, j = 0, 1, ..., J\}.$$

We define the levels **0**, **1**, **2**, ..., as the set of the states $\mathbf{0} = \{(0, j), j = 0, 1, ..., J\}$, $\mathbf{1} = \{(1, j), j = 0, 1, ..., j+1$, and $\boldsymbol{i} = \{(i, j), j = 0, 1, ..., J\}$ if $i \geq 2$, where the elements of the sets are arranged in lexicographical order. Using elementary arguments, the process $\{(N(t), L(t)), t \geq 0\}$ has a transition rate matrix $\boldsymbol{Q}$ which has a block-tridiagonal structure given by

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{B}_{00} & \boldsymbol{B}_{01} & & & & & \\ \boldsymbol{B}_{10} & \boldsymbol{B}_{11} & \boldsymbol{B}_{12} & & & & \\ & \boldsymbol{B}_{21} & \boldsymbol{A}_1 & \boldsymbol{A}_0 & & & \\ & & \boldsymbol{A}_2 & \boldsymbol{A}_1 & \boldsymbol{A}_0 & & \\ & & & \boldsymbol{A}_2 & \boldsymbol{A}_1 & \boldsymbol{A}_0 & \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot \end{bmatrix}.$$

Matrix $\boldsymbol{Q}$ is an infinitesimal generator of the Markov process $\{(N(t), L(t)), t \geq 0\}$ and is in the format of a quasi-birth-and-death (QBD) process. The sub-matrices $\boldsymbol{A}_0$, $\boldsymbol{A}_1$, and $\boldsymbol{A}_2$ are of order $(J+1) \times (J+1)$. They are given by $\boldsymbol{A}_0 = \lambda \boldsymbol{I}$,

$$\boldsymbol{A}_1 = \begin{bmatrix} -(\lambda+\theta) & 0 & \cdots & 0 & \theta \\ 0 & -(\lambda+\theta) & \cdots & 0 & \theta \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -(\lambda+\theta) & \theta \\ 0 & 0 & \cdots & 0 & -(\lambda+\mu_1+\mu_2) \end{bmatrix},$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \mu_1+\mu_2 \end{bmatrix}$$

where $\boldsymbol{I}$ is a $(J+1) \times (J+1)$ identity matrix. The boundary sub-matrices are defined by

$$\boldsymbol{B}_{00} = \begin{bmatrix} -(\lambda+\theta) & \theta & 0 & \cdots & 0 & 0 \\ 0 & -(\lambda+\theta) & \theta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -(\lambda+\theta) & \theta \\ 0 & 0 & 0 & \cdots & 0 & -\lambda \end{bmatrix},$$

$$\boldsymbol{B}_{01}=\begin{bmatrix}\lambda & 0 & \cdots & 0 & 0 & 0\\ 0 & \lambda & \cdots & 0 & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \lambda & 0 & 0\\ 0 & 0 & \cdots & 0 & \lambda & 0\end{bmatrix},\quad \boldsymbol{B}_{10}=\begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ \mu_1 & 0 & \cdots & 0\\ \mu_2 & 0 & \cdots & 0\end{bmatrix},$$

$$\boldsymbol{B}_{11}=\begin{bmatrix}-(\lambda+\theta) & 0 & \cdots & 0 & \theta & 0\\ 0 & -(\lambda+\theta) & \cdots & 0 & \theta & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & -(\lambda+\theta) & \theta & 0\\ 0 & 0 & \cdots & 0 & -(\lambda+\mu_1) & 0\\ 0 & 0 & \cdots & 0 & 0 & -(\lambda+\mu_2)\end{bmatrix},$$

$$\boldsymbol{B}_{12}=\begin{bmatrix}\lambda & 0 & \cdots & 0 & 0\\ 0 & \lambda & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & \lambda & 0\\ 0 & 0 & \cdots & 0 & \lambda\\ 0 & 0 & \cdots & 0 & \lambda\end{bmatrix},\quad \boldsymbol{B}_{21}=\begin{bmatrix}0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & \mu_2 & \mu_1\end{bmatrix}$$

where $\boldsymbol{B}_{00}$ is a $(J+1)\times(J+1)$ matrix, $\boldsymbol{B}_{01}$ and $\boldsymbol{B}_{21}$ are $(J+1)\times(J+2)$ matrices, $\boldsymbol{B}_{10}$ and $\boldsymbol{B}_{12}$ are $(J+2)\times(J+1)$ matrices, $\boldsymbol{B}_{11}$ is a $(J+2)\times(J+2)$ matrix.

# 3 Steady-State Analysis

In this section, we first derive the condition for the system to reach a steady state. Then, we obtain the matrix-geometric solution for the steady-state probabilities. Based on the explicit expressions of these probabilities, we derive some performance measures.

## 3.1 Stability Condition

We now derive the condition for the system to reach a steady state. We define matrix $\boldsymbol{A}=\boldsymbol{A}_0+\boldsymbol{A}_1+\boldsymbol{A}_2$. It is readily known that $\boldsymbol{A}$ is an irreducible generator of a Markov process. Let $\boldsymbol{\pi}=(\pi_0,\pi_1,...,\pi_J)$ be the stationary probability vector of this Markov process. Then, $\boldsymbol{\pi}$ satisfy the linear equations: $\boldsymbol{\pi}\boldsymbol{A}=\boldsymbol{0}$ and $\boldsymbol{\pi}\boldsymbol{e}=1$, where $\boldsymbol{e}$ is a column vector of order $J+1$ with all its elements equal to one. Solving these equations, we have $\pi_i=0$, for $i=0,1,...,J-1$, and $\pi_J=1$.

By using Theorem 3.1.1 in [11], the condition $\boldsymbol{\pi}\boldsymbol{A}_0\boldsymbol{e}<\boldsymbol{\pi}\boldsymbol{A}_2\boldsymbol{e}$ is the necessary and sufficient condition for stability of the QBD process. After some routine manipulation, the stability condition becomes as

$$\rho=\frac{\lambda}{\mu_1+\mu_2}<1. \tag{1}$$

## 3.2 Matrix-geometric Solution

Let $N$ and $L$ be the stationary random variables for the system size and the status of the servers. We denote the stationary probability by

$$x_{ij} = P\{N = i, L = j\} = \lim_{t\to\infty} P\{N(t) = i, L(t) = j\}, \quad (i, j) \in \Omega.$$

Under the stability condition $\rho < 1$, the stationary probability vector $\boldsymbol{x}$ of the generator $\boldsymbol{Q}$ exists. This stationary probability vector $\boldsymbol{x}$ is partitioned as $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, ...)$, where $\boldsymbol{x}_0 = (x_{00}, x_{01}, ..., x_{0J})$, $\boldsymbol{x}_1 = (x_{10}, x_{11}, ..., x_{1J+1})$, and $\boldsymbol{x}_i = (x_{i0}, x_{i1}, ..., x_{iJ})$ for $i \geq 2$.

Based on the matrix-geometric solution method developed by Neuts [11], the stationary probability vector $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, ...)$ is given by

$$\boldsymbol{x}_0\boldsymbol{B}_{00} + \boldsymbol{x}_1\boldsymbol{B}_{10} = 0, \tag{2}$$

$$\boldsymbol{x}_0\boldsymbol{B}_{01} + \boldsymbol{x}_1\boldsymbol{B}_{11} + \boldsymbol{x}_2\boldsymbol{B}_{21} = 0, \tag{3}$$

$$\boldsymbol{x}_1\boldsymbol{B}_{12} + \boldsymbol{x}_2(\boldsymbol{A}_1 + \boldsymbol{R}\boldsymbol{A}_2) = 0, \tag{4}$$

$$\boldsymbol{x}_i = \boldsymbol{x}_2\boldsymbol{R}^{i-2}, \qquad i = 3, 4, 5, ... \tag{5}$$

and the normalizing equation

$$\boldsymbol{x}_0\boldsymbol{e} + \boldsymbol{x}_1\boldsymbol{e}_1 + \boldsymbol{x}_2(\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{e} = 1 \tag{6}$$

where $\boldsymbol{e}_1$ is a column vector of order $J+2$ with all their elements equal to one, and matrix $\boldsymbol{R}$, called the rate matrix, is the minimal non-negative solution of the matrix quadratic equation

$$\boldsymbol{R}^2\boldsymbol{A}_2 + \boldsymbol{R}\boldsymbol{A}_1 + \boldsymbol{A}_0 = \boldsymbol{0}, \tag{7}$$

and the spectral radius of the rate matrix $\boldsymbol{R}$ is less than one.

In order to obtain the explicit expression of the rate matrix $\boldsymbol{R}$, we need to solve the matrix quadratic equation (7). The next theorem gives the explicit expression of the rate matrix $\boldsymbol{R}$. The proof of Theorem 1 is omitted.

**Theorem 1**. If $\rho < 1$, the matrix Eq. (7) has the minimal non-negative solution as follows:

$$\boldsymbol{R} = \begin{bmatrix} \frac{\lambda}{\lambda+\theta}\boldsymbol{I} & \rho\boldsymbol{e} \\ \boldsymbol{0} & \rho \end{bmatrix} \tag{8}$$

where $\boldsymbol{I}$ is a $(J+1)\times(J+1)$ identity matrix, and $\boldsymbol{e}$ is a column vector of order $J+1$ with all its elements equal to one.

Based on an explicit expression of the rate matrix $\boldsymbol{R}$ given by Theorem 1, we can solve Eqs. (2), (3), (4) and (6) to obtain the explicit expressions of the boundary probability vectors $\boldsymbol{x}_0$, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ under the stationary condition $\rho < 1$. Defining that

$$f = \frac{1}{\lambda+\theta}\left[1 - \left(\frac{\theta}{\lambda+\theta}\right)^J\right], \tag{9}$$

$$g = \frac{1}{\lambda} + \left[1 + \frac{\lambda}{\theta(1-\rho)}\right] f \tag{10}$$

and

$$h = \frac{\rho(1+\mu_1 f)}{1+\rho(1-\mu_1 f)}. \tag{11}$$

**Theorem 2**. If $\rho < 1$, the elements of the boundary probability vectors $\boldsymbol{x}_0$, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are given by

$$x_{00} = \frac{\mu_1}{\lambda+\theta}(1+h)K, \tag{12}$$

$$x_{0j} = \left(\frac{\theta}{\lambda+\theta}\right)^j x_{00}, \quad j = 0,1,...,J-1, \tag{13}$$

$$x_{0J} = \frac{\theta}{\lambda}\left(\frac{\theta}{\lambda+\theta}\right)^{J-1} x_{00}, \tag{14}$$

$$x_{1j} = \frac{\lambda}{\lambda+\theta}\left(\frac{\theta}{\lambda+\theta}\right)^j x_{00}, \quad j = 0,1,...,J-1, \tag{15}$$

$$x_{1J} = K, \quad x_{1J+1} = \frac{\mu_1}{\mu_2} hK, \tag{16}$$

$$x_{2j} = \left(\frac{\lambda}{\lambda+\theta}\right)^2 \left(\frac{\theta}{\lambda+\theta}\right)^j x_{00}, \quad j = 0,1,...,J-1, \tag{17}$$

$$x_{2J} = \left(1+\frac{\lambda}{\mu_2}\right) hK \tag{18}$$

where

$$K = \left[1+\mu_1 g(1+h)+\frac{1}{\mu_2}\left(\mu_1+\frac{\lambda+\mu_2}{1-\rho}\right)h\right]^{-1}. \tag{19}$$

**Proof**. Using the expression of $\boldsymbol{R}$ in Theorem 1, we can obtain the results of Theorem 2. The details of the proof are omitted. □

From Eq. (5), by using Theorem 1 and Theorem 2 we can obtain the probability vector $\boldsymbol{x}_i$, for $i \geq 3$. Define that

$$\phi_n = \sum_{j=1}^{n}\left(\frac{\lambda}{\lambda+\theta}\right)^{n-j} \rho^j, \quad n \geq 1. \tag{20}$$

**Theorem 3**. If $\rho < 1$, the elements of the probability vector $\boldsymbol{x}_i$ for $i \geq 3$ are given by

$$x_{ij} = \left(\frac{\lambda}{\lambda+\theta}\right)^i \left(\frac{\theta}{\lambda+\theta}\right)^j x_{00}, \quad j = 0,1,...,J-1, \tag{21}$$

$$x_{iJ} = \phi_{i-2}\frac{\lambda}{\lambda+\theta}\left[1-\left(\frac{\theta}{\lambda+\theta}\right)^J\right] x_{00} + \rho^{i-2} x_{2J} \tag{22}$$

where $x_{00}$ is given by Eq. (12), and $x_{2J}$ is given by Eq. (18).

**Proof**. Using Theorem 1, we obtain

$$\boldsymbol{R}^n = \begin{bmatrix} \left(\frac{\lambda}{\lambda+\theta}\right)^n \boldsymbol{I} & \phi_n \boldsymbol{e} \\ \boldsymbol{0} & \rho^n \end{bmatrix} \tag{23}$$

for $n \geq 1$, where $\phi_n$ is defined by Eq. (20). Substituting this expression into Eq. (5) yields $x_{ij}$ in Eqs. (21) and (22) for $i \geq 3$ and $j = 0, 1, ..., J$. □

### 3.3 Performance Measures

Let $x_i = P(L = i)$, $i = 0, 1, 2...$, be the stationary probability that there are $i$ customers in the system. The following theorem gives the stationary distribution of the system size.

**Theorem 4**. If $\rho < 1$, the distribution of the system size is given by

$$x_0 = \left(1 + \frac{\theta}{\lambda}\right) x_{00}, \tag{24}$$

$$x_1 = \frac{1}{\rho} x_{2J}, \tag{25}$$

$$x_2 = \frac{\lambda}{\lambda+\theta}\left[1 - \left(\frac{\theta}{\lambda+\theta}\right)^J\right] x_{00} + x_{2J}, \tag{26}$$

$$x_i = \left[\left(\frac{\lambda}{\lambda+\theta}\right)^{i-2} + \phi_{i-2}\right] \frac{\lambda}{\lambda+\theta}\left[1 - \left(\frac{\theta}{\lambda+\theta}\right)^J\right] x_{00} + \rho^{i-2} x_{2J}, \quad i = 3, 4, ... \tag{27}$$

where $x_{00}$ is given by Eq. (12), and $x_{2J}$ is given by Eq. (18).

**Proof**. The results of Theorem 4 are obtained by using Theorem 2 and Theorem 3. The details of the proof are omitted. □

**Corollary 1**. If $\rho < 1$, the mean system size is given by

$$E(N) = \frac{\lambda}{\theta(1-\rho)}\left(2 + \frac{\lambda}{\theta} + \frac{\rho}{1-\rho}\right)\left[1 - \left(\frac{\theta}{\lambda+\theta}\right)^J\right] x_{00} + \frac{1}{\rho(1-\rho)} x_{2J} \tag{28}$$

where $x_{00}$ is given by Eq. (12), and $x_{2J}$ is given by Eq. (18).

**Proof**. The result of Corollary 1 can be obtained by using Theorem 4. The details of the proof are omitted. □

Let $N_V$ be the number of vacations that the servers have taken under the condition that the servers are in vacation. The next theorem shows that $N_V$ follows the truncated geometric distribution. The proof of next theorem is omitted.

**Theorem 5**. If $\rho < 1$, then the number of vacations that the servers take continuously under the condition that the servers are in vacation follows the truncated geometric distribution as follows:

$$P(N_V = j+1) = \frac{\frac{\lambda}{\lambda+\theta}\left(\frac{\theta}{\lambda+\theta}\right)^j}{1-\left(\frac{\theta}{\lambda+\theta}\right)^J}, \quad j = 0,1,...,J-1. \tag{29}$$

# 4 Conditional Stochastic Decomposition

In an M/M/c queueing system with server vacations, the conditional stochastic decomposition property for the steady state queue length and the steady state waiting time has been established (see [12]). This property is under the condition that all servers are busy. We prove below that this property also holds for the system studied in this paper.

Let $Q_c = \{N-2|N \geq 2, L = J\}$ and $W_c = \{W|N \geq 2, L = J\}$ represent the conditional queue length and the conditional waiting time, respectively, given that both servers are busy. The next two theorems establish the conditional stochastic decomposition property on steady state queue length and waiting time. The proofs of these two theorems are omitted.

**Theorem 6**. If $\rho < 1$, the conditional queue length $Q_c$ can be decomposed into the sum of two independent random variables $Q_0$ and $Q_d$ as follows:

$$Q_c = Q_0 + Q_d \tag{30}$$

where $Q_0$ is the conditional queue length in a non-vacation M/M/2 system with two heterogeneous servers, and $Q_d$ is the additional queue length due to the vacations taken by servers. $Q_0$ has a geometric distribution with parameter $\rho$, and $Q_d$ has a probability generating function as follows:

$$Q_d(z) = \frac{1}{\sigma}\left\{x_{2J} + \frac{\lambda\rho z}{\theta+\lambda(1-z)}\left[1-\left(\frac{\theta}{\lambda+\theta}\right)^J\right]x_{00}\right\} \tag{31}$$

where

$$\sigma = x_{2J} + \rho\frac{\lambda}{\theta}\left[1-\left(\frac{\theta}{\lambda+\theta}\right)^J\right]x_{00}, \tag{32}$$

$x_{00}$ is given by Eq. (12), and $x_{2J}$ is given by Eq. (18).

**Theorem 7**. If $\rho < 1$, then the conditional waiting time $W_c$ can be decomposed into the sum of two independent random variables $W_0$ and $W_d$ as follows:

$$W_c = W_0 + W_d \tag{33}$$

where $W_0$ is the conditional waiting time in a non-vacation M/M/2 system with two heterogeneous servers, and $W_d$ is the additional delay due to the server vacations. $W_0$ follows an exponential distribution with parameter $(1-\rho)\times(\mu_1+\mu_2)$, and $W_d$ has the Laplace-Stieltjes transform as follows:

$$W_d(s) = \frac{1}{\sigma}\left\{x_{2J} + \frac{(\mu_1+\mu_2)\lambda\rho}{s(\lambda+\theta)+(\mu_1+\mu_2)\theta}\left[1-\left(\frac{\theta}{\lambda+\theta}\right)^J\right]x_{00}\right\} \tag{34}$$

where $\sigma$ is defined in Eq. (32), $x_{00}$ is given by Eq. (12), and $x_{2J}$ is given by Eq. (18).

# 5 Conclusions

We have investigated an M/M/2 queueing system with two heterogeneous servers under a variant vacation policy. We have provided the explicit expressions of the stationary distribution of the system size and the mean system size. We have also showed that the number of vacations that the servers take continuously under the condition that the servers are in vacation follows a truncated geometric distribution. The conditional stochastic decomposition properties of the queue length and the waiting time have been established for such a system.

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