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Approximate Fenchel-Lagrangian Duality for Constrained Set-Valued Optimization Problems

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Abstract In this article, we construct a Fenchel-Lagrangian ε-dual problem for set-valued optimization problems by using the perturbation methods. Some relationships between the solutions of the primal and the dual problems are discussed. Moreover, an ε-saddle point theorem is proved.

Keywords Set-valued optimization; ε-conjugate map; ε-weak efficiency; ε-weak saddle point

1 Introduction

In recent years, the vector optimization problems with set-valued maps have been investigated by many authors. There are many papers discussing the existence results and optimality conditions for set-valued vector optimization problems (see, for instance [1-5]).

Duality for set-valued vector optimization problems is an important class of duality theory. The Lagrangian duality for set-valued vector optimization problems was studied by Li and Chen [6] and Song [7]. The conjugate duality for set-valued vector optimization problems has been made in [8-13]. Recently, Li et al. [14] constructed three dual models for a set-valued vector optimization problem with explicit constrains by using the method of perturbation functions.

On the other hand, many researchers have focused on investigating the approximate solutions of set-valued optimization problems. For example, Vlyi [15] introduced some concepts of approximate solutions and presented an approximate saddle point theorem. Rong and Wu [16] gave some ε-weak saddle point theorem and ε-duality results by using the Lagrangian map. Jia and Li [17] introduced the concept of ε-conjugate map for set-valued map, constructed an ε-conjugate duality problem for set-valued vector optimization problem and proved some duality results.

Motivated by the work reported in [14, 16, 17], in this paper we will propose Fenchel-Lagrangian ε-dual model for a constraint set-valued vector optimization problem by using the perturbation methods and derive some duality results and an ε-weak saddle point theorem.

2 Preliminaries

Let X be a real topological vector space, Y be a real topological vector space which is partially ordered by a pointed closed convex cone K with nonempty interior int$K$ in Y. We denote by Y^* the topological dual space of Y. For a subset $A \subset Y$, we define the dual cone of A by $A^* = \{y^* \in Y^* : \langle y^*, y \rangle \geq 0, \ \forall y \in A\}$. For any $x, y \in Y$, we define the following ordering relations:

$x < y \Leftrightarrow y - x \in \ \mathrm{int}K$, and $x \not< y \Leftrightarrow y - x \notin \ \mathrm{int}K$.

Let $B \subset Y$ be a nonempty subset and $\varepsilon \in K$. The set $\mathrm{Wmin}_\varepsilon(B)$ of all ε-weak minimal point and the set $\mathrm{Wmax}_\varepsilon(B)$ of all ε-weak maximal point of B are defined by (see [16])

$$\mathrm{Wmin}_\varepsilon(B) = \{b \in B : y + \varepsilon \not< b, \forall y \in B\} \text{ and } \mathrm{Wmax}_\varepsilon(B) = \{b \in B : b \not< y - \varepsilon, \forall y \in B\}$$

respectively. Clearly, $\mathrm{Wmin}_\varepsilon(-B) = -\mathrm{Wmax}_\varepsilon(B)$, and $\mathrm{Wmax}_\varepsilon(-B) = -\mathrm{Wmin}_\varepsilon(B)$.

Let F be a set-valued map from X to Y, $A \subset X$. We denote $\mathrm{dom}F = \{x \in X : F(x) \neq \emptyset\}$ and $F(A) = \bigcup_{x \in A} F(x)$.

Proposition 2.1. ([17]) Let F_1 and F_2 be set-valued maps from X to Y. Then

$$\mathrm{Wmax}_\varepsilon \bigcup_{x \in X} [F_1(x) + F_2(x)] \subset \mathrm{Wmax}_\varepsilon \bigcup_{x \in X} [F_1(x) + \mathrm{Wmax}_\varepsilon F_2(x)]. \tag{1}$$

Further, if we assume that $F_2(x) \subset \mathrm{Wmax}_\varepsilon F_2(x) - K, \forall x \in X$, then the (1) becomes equality.

Let $L(X,Y)$ be the space of all linear continuous operators from X to Y.

Definition 2.1.([17]) A set-valued map $F^* : L(X,Y) \to 2^Y$ defined by

$$F^*(T) = \mathrm{Wmax}_\varepsilon \bigcup_{x \in X} [T(x) - F(x)], \ \ \forall T \in L(X,Y),$$

is called the ε-conjugate map of F.

3 ε-dual problem

Let X be a real topological vector space, Y and Z be two real partially ordered topological vector spaces, $K \subset Y$ and $E \subset Z$ be two pointed closed convex cones with $\mathrm{int}K \neq \emptyset$ and $\mathrm{int}E \neq \emptyset$. We define a subset $L^+(Z,Y)$ of $L(Z,Y)$ as $L^+(Z,Y) = \{\Lambda \in L(Z,Y) : \Lambda(E) \subset K\}$. Let $F : X \to 2^Y$ and $G : X \to 2^Z$ be two set-valued maps with $\mathrm{dom}(F) \neq \emptyset$. Let S be a subset of X with $S \subset \mathrm{dom}(F)$. We consider the following set-valued optimization problem

$$(P) \quad \min_{x \in \Omega} F(x),$$

where $\Omega = \{x \in S : G(x) \cap (-E) \neq \emptyset\}$. We always assume that the feasible set $\Omega \neq \emptyset$.

Definition 3.1. A feasible solution $x \in \Omega$ is said to be an ε-weak minimal solution of the problem (P) if

$$F(x) \cap \mathrm{Wmin}_\varepsilon(F(\Omega)) \neq \emptyset.$$

In the following, we will construct the a Fenchel-Lagrangian ε-dual model for (P) by using the perturbation methods. We first give the Fenchel-Lagrangian map $\phi^{FL} : X \times X \times Z \to 2^Y$ as

$$\phi^{FL}(x,p,q) = \begin{cases} F(x+p), & x \in S, G(x) \cap (-E-q) \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$

By Definition 2.1, we can easily obtain the follows.

$$-(\phi^{FL})^*(0,\Gamma,\Lambda) = \mathrm{Wmin}_\varepsilon\{\bigcup_{r\in X}[-\Gamma(r)+F(r)] + \bigcup_{x\in S}[\Gamma(x)+\Lambda(G(x))]+\Lambda(E)\}$$

for any $\Gamma \in L(X,Y)$ and $\Lambda \in L^+(Z,Y)$.

Now we define the Fenchel-Lagrangian ε-dual problem as follows

$$(D^{FL}) \quad \max \bigcup_{\substack{\Gamma\in L(X,Y)\\ \Lambda\in L^+(Z,Y)}} \Big\{\mathrm{Wmin}_\varepsilon\{\bigcup_{r\in X}[-\Gamma(r)+F(r)] + \bigcup_{x\in S}[\Gamma(x)+\Lambda(G(x))]+\Lambda(E)\}\Big\}.$$

Definition 3.2. $(\bar{\Gamma},\bar{\Lambda}) \in L(X,Y)\times L^+(Z,Y)$ is said to be an ε-weak maximal solution of (D^{FL}), if

$$-(\phi^{FL})^*(0,\bar{\Gamma},\bar{\Lambda}) \cap \mathrm{Wmax}_\varepsilon\Big(\bigcup_{\substack{\Gamma\in L(X,Y)\\ \Lambda\in L^+(Z,Y)}} -(\phi^{FL})^*(0,\Gamma,\Lambda)\Big) \neq \emptyset.$$

Theorem 3.1. (ε-Weak duality) For any $x \in \Omega$, $\Gamma \in L(X,Y)$ and $\Lambda \in L^+(Z,Y)$, we have that

$$F(x)\bigcap\Big(\mathrm{Wmin}_\varepsilon\Big[\bigcup_{r\in X}[-\Gamma(r)+F(r)] + \bigcup_{x\in S}[\Gamma(x)+\Lambda(G(x))]+\Lambda(E)\Big]-\varepsilon-\mathrm{int}K\Big) = \emptyset.$$

Proof. Suppose to the contrary. Then there exist $\bar{x}\in\Omega$, $\bar{\Gamma}\in L(X,Y)$ and $\bar{\Lambda}\in L^+(Z,Y)$ such that

$$F(\bar{x})\bigcap\Big(\mathrm{Wmin}_\varepsilon[\bigcup_{r\in X}(-\bar{\Gamma}(r)+F(r)) + \bigcup_{x\in S}(\bar{\Gamma}(x)+\bar{\Lambda}(G(x)))+\bar{\Lambda}(E)]-\varepsilon-\mathrm{int}K\Big) \neq \emptyset.$$

Thus there exist $\bar{y}_1 \in F(\bar{x})$ and

$$\bar{y}_2 \in \mathrm{Wmin}_\varepsilon[\bigcup_{r\in X}[-\bar{\Gamma}(r)+F(r)] + \bigcup_{x\in S}[\bar{\Gamma}(x)+\bar{\Lambda}(G(x))]+\bar{\Lambda}(E)] \tag{2}$$

such that

$$\bar{y}_1 \in \bar{y}_2 - \varepsilon - \mathrm{int}K. \tag{3}$$

Then, we have

$$\begin{aligned}&\bar{y}_2 - (\bar{y}_1 - \bar{\Gamma}(\bar{x}) + \bar{\Gamma}(\bar{x}) + \bar{\Lambda}(q) + \bar{\Lambda}(0)) - \varepsilon\\ &= \bar{y}_2 - \bar{y}_1 - \varepsilon + \bar{\Lambda}(-q)\\ &\in \mathrm{int}K + \bar{\Lambda}(-q)\\ &\subset \mathrm{int}K + K = \mathrm{int}K\end{aligned}$$

for any $q \in G(\bar{x})\cap(-E)$. This contradicts (2). □

Now we discuss the strong duality between the primal (P) and the dual problem (D^{FL}). First, we define the set-valued map $W : X\times Z \to 2^Y$ as

$$W(p,q) = \mathrm{Wmin}_\varepsilon \bigcup_{x\in X} \phi^{FL}(x,p,q).$$

It is obvious that $W(0,0) = \mathrm{Wmin}_\varepsilon \bigcup_{x\in\Omega} F(x)$.

Definition 3.3. Let $\bar{p} \in X$, $\bar{q} \in Z$ and $\bar{z} \in W(\bar{p},\bar{q})$. $(\Gamma,\Lambda) \in L(X,Y)\times L^+(Z,Y)$ is said to be the positive ε-subgradient of W at $(\bar{p},\bar{q},\bar{z})$, if

$$\bar{z} - \Gamma(\bar{p}) - \Lambda(\bar{q}) \in \mathrm{Wmin}_\varepsilon \bigcup_{p\in X, q\in Z} [W(p,q) - \Gamma(p) - \Lambda(q)].$$

The set of all positive ε-subgradient of W at $(\bar{p},\bar{q},\bar{z})$ is called the ε-subdifferential of W at $(\bar{p},\bar{q},\bar{z})$ and is denoted by $\partial_\varepsilon^+(\bar{p},\bar{q},\bar{z})$.

Definition 3.4. The problem (P) is said to be stable with respect to ϕ^{FL}, if

$$\partial_\varepsilon^+ W((0,0),z) \neq \emptyset, \ \ \forall z \in W(0,0).$$

Theorem 3.2. Let the problem (P) be stable with respect to ϕ^{FL}, $\bar{x}$ be an ε-weak minimal solution of (P) and $\bigcup_{x\in X} \phi^{FL}(x,p,q) \subset W(p,q)+K$, $\forall (p,q) \in X\times Z$. Then there exist $\bar{\Gamma} \in L(X,Y)$, $\bar{\Lambda} \in L^+(Z,Y)$ to be the ε-weak maximal solution of (D^{FL}) such that

$$F(\bar{x})\bigcap \mathrm{Wmin}_\varepsilon \Big[\bigcup_{r\in X}[-\bar{\Gamma}(r)+F(r)] + \bigcup_{x\in S}[\bar{\Gamma}(x)+\bar{\Lambda}(G(x))] + \bar{\Lambda}(E)\Big] \neq \emptyset. \tag{4}$$

Proof. Since $\bar{x}$ is an ε-weak minimal solution of (P), we have that $\bar{x} \in S$, $G(\bar{x}) \cap (-E) \neq \emptyset$ and $\exists \bar{y} \in F(\bar{x})$ such that

$$\bar{y} \in \mathrm{Wmin}_\varepsilon \bigcup_{x\in\Omega} F(x) = \mathrm{Wmin}_\varepsilon \bigcup_{x\in X} \phi^{FL}(x,0,0) = W(0,0).$$

The stability of (P) implies that $\partial_\varepsilon^+ W((0,0),\bar{y}) \neq \emptyset$. Then there exist $\bar{\Gamma} \in L(X,Y)$ and $\bar{\Lambda} \in L^+(Z,Y)$ such that

$$\bar{y} = \bar{y} - \bar{\Gamma}(0) - \bar{\Lambda}(0) \in \mathrm{Wmin}_\varepsilon \bigcup_{p\in X, q\in Z} [W(p,q) - \bar{\Gamma}(p) - \bar{\Lambda}(q)] = -W^*(\bar{\Gamma},\bar{\Lambda}).$$

Since $\bigcup_{x\in X} \phi^{FL}(x,p,q) \subset W(p,q)+K$, $\forall (p,q) \in X\times Z$, from Proposition 2.1 we have that

$$W^*(\Gamma,\Lambda) = (\phi^{FL})^*(0,\Gamma,\Lambda), \forall (\Gamma,\Lambda) \in L(X,Y)\times L^+(Z,Y),$$

and so $\bar{y} \in -(\phi^{FL})^*(0,\bar{\Gamma},\bar{\Lambda})$. Therefore, (4) is true.

On the other hand, we can show that $(\bar{\Gamma},\bar{\Lambda})$ is the ε-weak maximal solution of (D^{FL}). For any $y \in \bigcup_{\substack{\Gamma\in L(X,Y)\\ \Lambda\in L^+(Z,Y)}} \Big\{-(\phi^{FL})^*(0,\Gamma,\Lambda)\Big\}$, there exists $(\tilde{\Gamma},\tilde{\Lambda}) \in L(X,Y)\times L^+(Z,Y)$ such that

$$y \in -(\phi^{FL})^*(0,\tilde{\Gamma},\tilde{\Lambda}) = -W^*(\tilde{\Gamma},\tilde{\Lambda}).$$

Since $W^*(\tilde{\Gamma},\tilde{\Lambda}) = \mathrm{Wmax}_\varepsilon \bigcup_{p\in X, q\in Z}[\tilde{\Gamma}(p)+\tilde{\Lambda}(q)-W(p,q)]$ and $-\bar{y} \in -W(0,0)$, we have that

$$-y \not< -\bar{y} - \varepsilon,$$

which is equivalent to $\bar{y} \not< y - \varepsilon$. Thus $(\bar{\Gamma},\bar{\Lambda})$ is an ε-weak maximal solution of (D^{FL}). □

Theorem 3.3. Let $(\bar{x},\bar{\Gamma},\bar{\Lambda}) \in \Omega \times L(X,Y) \times L^+(Z,Y)$. If there exists $\bar{y} \in Y$, such that

$$\bar{y} \in F(\bar{x}) \bigcap \mathrm{Wmin}_\varepsilon \Big[\bigcup_{r\in X} [-\bar{\Gamma}(r) + F(r)] + \bigcup_{x\in S} [\bar{\Gamma}(x) + \bar{\Lambda}(G(x))] + \bar{\Lambda}(E) \Big]. \tag{5}$$

Then $\bar{x}$ is an ε-weak minimal solution of (P) and $(\bar{\Gamma},\bar{\Lambda})$ is an ε-weak maximal solution of (D^{FL}).

Proof. From Theorem 3.1, we have that, for any $(x,\Gamma,\Lambda) \in \Omega \times L(X,Y) \times L^+(Z,Y)$,

$$F(x) \bigcap \Big(\mathrm{Wmin}_\varepsilon \Big[\bigcup_{r\in X} [-\Gamma(r) + F(r)] + \bigcup_{x\in S} [\Gamma(x) + \Lambda(G(x))] + \Lambda(E) \Big] - \varepsilon - \mathrm{int}K \Big) = \emptyset. \tag{6}$$

If $\bar{x}$ is not an ε-weak minimal solution of (P), then there exists $\tilde{y} \in \bigcup_{x\in\Omega} F(x)$, such that $\tilde{y} + \varepsilon < \bar{y}$, which together with (5) shows that

$$\tilde{y} \in \bar{y} - \varepsilon - \mathrm{int}K \subset \mathrm{Wmin}_\varepsilon \Big[\bigcup_{r\in X} [-\bar{\Gamma}(r) + F(r)] + \bigcup_{x\in S} [\bar{\Gamma}(x) + \bar{\Lambda}G(x)] + \bar{\Lambda}(E) \Big] - \varepsilon - \mathrm{int}K.$$

This contradicts (6). Hence $\bar{x}$ is an ε-weak minimal solution of (P).

If $(\bar{\Gamma},\bar{\Lambda})$ is not an ε-weak maximal solution of (D^{FL}), then there exists$(\tilde{\Gamma},\tilde{\Lambda}) \in L(X,Y) \times L^+(Z,Y)$ with $\tilde{z} \in \mathrm{Wmin}_\varepsilon \Big[\bigcup_{r\in X} [-\tilde{\Gamma}(r) + F(r)] + \bigcup_{x\in S} [\tilde{\Gamma}(x) + \tilde{\Lambda}G(x)] + \tilde{\Lambda}(E) \Big]$ such that $\bar{y} < \tilde{z} - \varepsilon$. Hence,

$$\bar{y} \in \tilde{z} - \varepsilon - \mathrm{int}K \subset \mathrm{Wmin}_\varepsilon \Big[\bigcup_{r\in X} [-\tilde{\Gamma}(r) + F(r)] + \bigcup_{x\in S} [\tilde{\Gamma}(x) + \tilde{\Lambda}G(x)] + \tilde{\Lambda}(E) \Big] - \varepsilon - \mathrm{int}K.$$

This contradicts (6). Hence $(\bar{\Gamma},\bar{\Lambda})$ is an ε-weak maximal solution of (D^{FL}). □

4 Lagrangian map and saddle point

In this section, we introduce a Lagrangian map for (P), which is different from that in [16], and propose an ε-saddle point theorem.

Definition 4.1. The set-valued map $L : S \times L(X,Y) \times L^+(Z,Y) \to 2^Y$, defined by

$$L(x,\Gamma,\Lambda) = \bigcup_{r\in X} [-\Gamma(r) + F(r)] + \Gamma(x) + \Lambda(G(x)) + \Lambda(E)$$

is called the Lagrangian map of the problem (P) relative to the perturbation map ϕ^{FL}.

Definition 4.2. A point $(\bar{x},\bar{\Gamma},\bar{\Lambda}) \in S \times L(X,Y) \times L^+(Z,Y)$ is called an ε-saddle point of $L(x,\Gamma,\Lambda)$, if

$$L(\bar{x},\bar{\Gamma},\bar{\Lambda}) \cap \mathrm{Wmax}_\varepsilon \bigcup_{\substack{\Gamma\in L(X,Y)\\ \Lambda\in L^+(Z,Y)}} L(\bar{x},\Gamma,\Lambda) \cap \mathrm{Wmin}_\varepsilon \bigcup_{x\in S} L(x,\bar{\Gamma},\bar{\Lambda}) \neq \emptyset.$$

Theorem 4.1. A point $(\bar{x},\bar{\Gamma},\bar{\Lambda}) \in S \times L(X,Y) \times L^+(Z,Y)$ is an ε-saddle point of Lagrangian map $L(x,\Gamma,\Lambda)$ if and only if there exist $\bar{y} \in \bigcup_{r\in X} [-\bar{\Gamma}(r) + F(r)]$ and $\bar{z} \in G(\bar{x}) + E$ such that the following conditions hold:

(a) $\bar{y}+\bar{\Gamma}(\bar{x})+\bar{\Lambda}(\bar{z}) \in \mathrm{Wmin}_{\varepsilon} \bigcup_{x\in S} L(x,\bar{\Gamma},\bar{\Lambda})$;

(b) $-\bar{\Lambda}(\bar{z}) \in K\backslash(\varepsilon+\mathrm{int}K)$;

(c) $G(\bar{x})+E \subset -E$;

(d) $\left(\bigcup_{r\in X}[-\Gamma(r)+F(r)]+\Gamma(\bar{x})-\bar{y}-\bar{\Gamma}(\bar{x})-\bar{\Lambda}(\bar{z})-\varepsilon\right)\bigcap \mathrm{int}K=\emptyset,\ \forall\Gamma\in L(X,Y)$.

Proof. " $\Rightarrow$ " $(\bar{x},\bar{\Gamma},\bar{\Lambda}) \in S\times L(X,Y)\times L^{+}(Z,Y)$ is an ε-saddle point of the Lagrangian map $L(x,\Gamma,\Lambda)$. Then there exist $\bar{y}\in\cup_{r\in X}[-\bar{\Gamma}(r)+F(r)]$ and $\bar{z}\in G(\bar{x})+E$ such that

$$\bar{y}+\bar{\Gamma}(\bar{x})+\bar{\Lambda}(\bar{z}) \in \mathrm{Wmin}_{\varepsilon} \bigcup_{x\in S} L(x,\bar{\Gamma},\bar{\Lambda}), \tag{7}$$

$$\bar{y}+\bar{\Gamma}(\bar{x})+\bar{\Lambda}(\bar{z}) \in \mathrm{Wmax}_{\varepsilon} \bigcup_{\substack{\Gamma\in L(X,Y)\\ \Lambda\in L^{+}(Z,Y)}} L(\bar{x},\Gamma,\Lambda). \tag{8}$$

This shows that condition (a) is true and for all $\Gamma\in L(X,Y)$ and $\Lambda\in L^{+}(Z,Y)$,

$$\Big[\bigcup_{r\in X}[-\Gamma(r)+F(r)]+\Gamma(\bar{x})+\Lambda(G(\bar{x}))+\Lambda(E)-(\bar{y}+\bar{\Gamma}(\bar{x})+\bar{\Lambda}(\bar{z})+\varepsilon)\Big]\bigcap \mathrm{int}K=\emptyset. \tag{9}$$

Taking $\Gamma=\bar{\Gamma}$ in (9), we have

$$\Big[\bigcup_{r\in X}[-\bar{\Gamma}(r)+F(r)]+\Lambda(G(\bar{x}))+\Lambda(E)-(\bar{y}+\bar{\Lambda}(\bar{z})+\varepsilon)\Big]\bigcap \mathrm{int}K=\emptyset,\ \ \forall\Lambda\in L^{+}(Z,Y).$$

Since $\bar{y}\in\cup_{r\in X}[-\bar{\Gamma}(r)+F(r)]$, we have that

$$\Lambda(z)-\bar{\Lambda}(\bar{z})-\varepsilon\notin \mathrm{int}K,\ \ \forall\Lambda\in L^{+}(Z,Y),\ \forall z\in G(\bar{x})+E. \tag{10}$$

Suppose that $-\bar{z}\notin E$. Since the convex cone E is closed, we have that $E=E^{**}$. Hence, there exists $\bar{\lambda}\in E^{*}$, such that $\langle\bar{z},\bar{\lambda}\rangle>0$. For any fixed $\tilde{k}\in\mathrm{int}K$, we define a map $\tilde{\Lambda}:Z\to Y$ as

$$\tilde{\Lambda}(z)=\frac{\langle z,\bar{\lambda}\rangle}{\langle\bar{z},\bar{\lambda}\rangle}(\tilde{k}+\varepsilon)+\bar{\Lambda}(z).$$

We can easy see that $\tilde{\Lambda}\in L^{+}(Z,Y)$ and $\tilde{\Lambda}(\bar{z})-\bar{\Lambda}(\bar{z})-\varepsilon=\tilde{k}\in\mathrm{int}K$, which contradicts (10). Hence, $-\bar{z}\in E$ and so $-\bar{\Lambda}(\bar{z})\in K$. Taking $\Lambda=0$ in (10), we have $-\bar{\Lambda}(\bar{z})-\varepsilon\notin\mathrm{int}K$. Therefore $-\bar{\Lambda}(\bar{z})\in K\backslash(\varepsilon+\mathrm{int}K)$.

Next, we will prove that $G(\bar{x})+E\subset -E$. Suppose to the contrary that there exists $z_0\in G(\bar{x})+E$ such that $-z_0\notin E$. We can find $\lambda_0\in E^{*}$ such that $\langle\lambda_0,z_0\rangle>0$. Taking any fixed $k_0\in\mathrm{int}K$, let $\Lambda_0(z)=\frac{\langle z,\lambda_0\rangle}{\langle z_0,\lambda_0\rangle}(k_0+\varepsilon)$. Obviously, $\Lambda_0\in L^{+}(Z,Y)$ and $\Lambda_0(z_0)-\bar{\Lambda}(\bar{z})-\varepsilon=k_0-\bar{\Lambda}(\bar{z})\in\mathrm{int}K+K=\mathrm{int}K$, which contradicts (10). Therefore, $G(\bar{x})+E\subset -E$.

Taking $\Lambda=0$ in (9), we have that condition (d) holds.

" $\Leftarrow$ " From condition (d), we have that

$$y+\Gamma(\bar{x})-(\bar{y}+\bar{\Gamma}(\bar{x})+\bar{\Lambda}(\bar{z})+\varepsilon)\notin\mathrm{int}K,\ \ \forall\Gamma\in L(X,Y),\ \forall y\in\cup_{r\in X}[-\Gamma(r)+F(r)].$$

Condition (c) shows that $-\Lambda(z) \in K$ for any $z \in G(\bar{x}) + E$ and $\Lambda \in L^+(Z,Y)$. Then, one can easy obtain that for all $\Gamma \in L(X,Y)$ and $\Lambda \in L^+(Z,Y)$,

$$y + \Gamma(\bar{x}) + \Lambda(z) - (\bar{y} + \bar{\Gamma}(\bar{x}) + \bar{\Lambda}(\bar{z}) + \varepsilon) \notin \text{int}K, \ \ \forall y \in \bigcup_{r \in X} [-\Gamma(r) + F(r)], \ \forall z \in G(\bar{x}) + E.$$

That is to say

$$\bar{y} + \bar{\Gamma}(\bar{x}) + \bar{\Lambda}(\bar{z}) \in \text{Wmax}_{\varepsilon} \bigcup_{\substack{\Gamma \in L(X,Y) \\ \Lambda \in L^+(Z,Y)}} L(\bar{x}, \Gamma, \Lambda),$$

which together with condition (a) shows that $(\bar{x}, \bar{\Gamma}, \bar{\Lambda})$ is an ε-saddle point of the Lagrangian map $L(x, \Gamma, \Lambda)$. □

References

[1] H.W. Corley, Existence and Lagrange duality for maximization of set-valued functions, J. Optim. Theory Appl. 54 (1987), 489-501.

[2] H. W. Corley, Optimality conditions for Maximizations of set-valued functions, J. Optim. Theory Appl. 58 (1988), 1-10.

[3] P.Q. Khanh, L.M. Luu, Necessary optimality conditions in problems involving set-valued maps with parameters, ACTA Math. Vietnamica, 26 (2001), 279-295.

[4] G.Y. Chen, J. Jahn, Optimally conditions for set-valued optimization problems, Math. Meth. Oper. Res. 48 (1998), 187-200.

[5] N. Gadhi, L. Lafhim, Necessary optimality conditions for set-valued optimization problems via the extremal principle, Positivity, 13 (2009), 657-669.

[6] Z.F. Li, G.Y. Chen, Lagrangian multipliers, saddle points, and duality in vector optimization of set-valued maps, J. Math. Anal. Appl. 215 (1997), 297-316.

[7] W. Song, Lagrangian Duality for Minimization of Nonconvex Multifunctions. J. Optim. Theory Appl. 93 (1997), 167-182.

[8] T. Tanino, Y. Sawaragi, Conjugate maps and duality in multiobjective optimization, J. Optim. Theory Appl. 31 (1980), 473-499.

[9] T. Tanino, Conjugate duality in vector optimization. J. Math. Anal. Appl. 167 (1992), 84-97.

[10] W. Song, Conjugate duality in set-valued vector optimization, J. Math. Anal. Appl. 216 (1997) 265-283.

[11] W. Song, A generalization of Fenchel duality in set-valued vector optimization, Math. Meth. Oper. Res. 48 (1998), 259-272.

[12] A.Y. Azimov, Duality for Set-Valued Multiobjective Optimization Problems, Part 1: Mathematical Programming, J Optim Theory Appl 137 (2008), 61-74.

[13] S.J. Li, C.R. Chen, S.Y. Wu, Conjugate dual problems in constrained set-valued optimization and applications, Eur. J. Oper. Res. 196 (2009), 21-32.

[14] S. J. Li, X.K. Sun, H. M. Liu, S.F. Yao, K.L. Teo, Conjugate Duality in Constrained Set-Valued Vector Optimization, Numer. Funct. Anal. Optim. 32 (2011), 65-82.

[15] I. Valyi, Approximate saddle-point theorems in vector optimization, J. Optim. Theory Appl. 55 (1987), 435-448.

[16] W.D. Rong, Y.N. Wu, ε-weak minimal solutions of vector optimization problems with set-valued maps. J. Optim. Theory Appl. 106 (2000), 569-579.

[17] J.H. Jia, Z.F. Li, ε-Conjugate maps and ε-conjugate duality in vector optimization with set-valued maps, Optimization, 57 (2008), 621-633.