The Sixth International Symposium on Operations Research and Its Applications (ISORA'06)
Xinjiang, China, August 8–12, 2006

Ranking DEA Efficient Units with the Most Compromising Common Weights

Fuh-Hwa Franklin Liu* Hao-Hsuan Peng
Hong-Wei Chang

Department of Industrial Engineering & Management, National Chiao Tung University, Taiwan

Abstract One may employ Data Envelopment Analysis (DEA) to discriminate decision-making units (*DMU*s) into efficient and inefficient ones base upon the multiple inputs and output performance indices. In this paper we consider that there is a centralized decision maker (DM) who 'owns' or 'supervises' all the *DMU*s. In such intraorganizational scenario the DM has an interest in discriminating the efficient *DMU*s (*eDMU*s). This paper presents a new method that determines the most compromising set of weights for the indices'. The total of the new efficiency scores of *eDMU*s with the most compromising set of indices' weights has the least total gaps to the compromised datum. The *eDMU*s that have efficiency score equal to one are located on the datum. The other *eDMU*s are either located above or below the datum. The approach is analog to the ordinary least-square method (OLS) of the residuals in statistical regression analysis. We compare the results of an example with multiple inputs and single output under the proposed approach and regression analysis.

Keywords data envelopment analysis; common weight; ranking; regression analysis; compromising datum

1 Introduction

Charns, Cooper and Rhodes (CCR) (1978) introduce Data Envelopment Analysis DEA that assess the *comparative* or *relative* efficiency of homogeneous operating decision-making units (*DMU*s) such as schools, hospitals, or sales outlets. The assessment of a *DMU* uses a set of resources referred to as input indices which it transforms into a set of outcomes referred to as output indices. DEA deals with the ratio between weighted sum of outputs and the weighted sum of inputs. DEA discriminates *DMU*s into two categories: efficient *DMU*s (*eDMU*s) and inefficient *DMU*s (*iDMU*s). The relative efficiency of an *iDMU* is reference to a set of *eDMU*s. One cannot in general derive by means of DEA to have some *absolute* measure of efficiency unless he/she makes additional assumptions that the *DMU*s being compared include a 'sufficient' number of *DMU*s which are efficient in some absolute sense. Each *DMU* in the efficient category is assigned a set of weights of indices so that its

*Corresponding author. E-mail: fliu@mail.hinet.net

relative efficiency score is equal to one, the maximum. DEA cannot provide enough information to rank the *eDMU*s with the same measure 1. If one further wants to understand which the best is, he/she needs another indicator to discriminate among the *eDMU*s.

Research about the idea of common weights and rankings has developed gradually in recent years. Cook et al. (1990) first proposed the idea of common weights in DEA. Andersen and Petersen (1993) developed procedures for ranking only the efficient units in the DEA. Doyle and Green (1994) developed a rank scale method utilizing the cross-efficiency matrix by ranking the average efficiency ratios of each unit in the runs of all the other runs. Tofallis et al. (2001) combined data envelopment analysis and regression to efficiency assessment. Liu and Peng (2004a) proposed common weights analysis (CWA) model to rank *DMU*s in the category of efficient. CWA determines an implicit datum under the assumption that the maximum efficiency is equal 1 among the *eDMU*s. The efficiency of each *DMU* refer to the datum is less or equal to one. Liu and Peng (2004b) proposed super common weights analysis (SCWA) model to rank *DMU*s in the category of efficient. SCWA determines an implicit datum under the assumption that the minimum efficiency is equal 1 among the *eDMU*s. The efficiency of each *DMU* refer to the datum at least is equal to one.

There are, however, situations in which all the *DMU*s fall under the umbrella of a centralized decision-maker (DM) who oversees them. This type of situation occurs whenever all the units belong to the same organization (public or private) who provides the units with the resources necessary for obtaining their inputs. Many of the traditional applications of DEA such as bank branches, hospitals, university departments, police stations, etc., fall into this category. In this situation, the DM is interest to rank *eDMU*s that is respect to the set of most compromising indices' weights.

In this paper we propose a procedure to determine a set of weights for the indices that is the most compromising among the *eDMU*s. The *implicit* efficiency datum among the *eDMU*s is set to one. Employ the most compromising weights (*MCW*s), *eDMU*s' implicit efficiencies could be above, equal, or below the datum. The total of gaps of the implicit efficiencies to the datum has been minimized. We review the related literature such as: multiple regression analysis, data envelopment analysis, common weight analysis, and super-common weight analysis in the second section. In the third section, we develop the most compromising weights analysis (MCWA) model. Then, the MCWA ranking rules are explained. At the same time, the Two-Phase method is introduced to deal with the problem of possible alternative solutions in the linear programming. In the fourth section, we compare the results of an example with multiple inputs and single output under the proposed approach, regression analysis, CWA and SCWA. Finally, the conclusion and future development opportunities are suggested in the last paragraph.

2 Literature Review

2.1 Multiple Regression Analysis

The following equation is the expression of multiple linear regression without intercept model for m independent variables (indices) X_1, X_2, ..., X_m to a dependent variable (index) Y for DMU_j

$$y_j = \beta_1 x_{1j} + \beta_2 x_{2j} + ... + \beta_m x_{mj} + e_j^R. \tag{1}$$

The β_1,β_2, ..., $\beta_m$are called regression coefficients and e_j^R is called the residual or error term of the DMU_j. The associated data set may be arranged as the following table.

Table 1: Data Set of Multiple Regression

DMU	Input Indices				Output Index
	X_1	X_2	$\cdots$	X_m	Y
1	x_{11}	x_{21}	$\cdots$	x_{m1}	y_1
2	x_{12}	$\ddots$		x_{m2}	y_2
$\vdots$	$\vdots$			$\vdots$	$\vdots$
n	x_{1n}	x_{2n}	$\cdots$	x_{mn}	y_n

As shown in many regression text books, the problem of minimizing the sum of squares of these$n$ residuals would find $\hat{\beta}_1$, $\hat{\beta}_2$, ...,$\hat{\beta}_m$, the least squares estimates of β_1, β_2, ..., β_m or simply least squares regression coefficients. The fitted regression model is then

$$Y = \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 + \ldots + \hat{\beta}_m X_m \tag{2}$$

gives a point estimate of the mean of Y for a particular data set as depicted in the above table.

2.2 DEA Framework

Cooper et al. (1978) proposes that there are n DMUs to be assessed with m inputs and s outputs indices. For each DMU, say DMU_j, the given values on the indices are denoted as (x_{1j}, x_{2j}, ..., x_{mj}) and (y_{1j}, y_{2j}, ..., y_{sj}), respectively. Given the data, we measure the efficiency of each DMU once and hence need n optimizations, one for each DMU to be evaluated. Let the DMU_j being evaluated on any trial be designated as DMU_o where o ranges over 1,2,...,n. We solve the following fractional programming problem (P1) to obtain the *relative* input "weights" (v_{io}, i = 1,2,...,m) and the *relative* output "weights" (u_{ro}, r = 1,2,...,s) as variables. ε is a positive Archimedean infinitesimal constant.

(P1) DEA-FP

$$\max \quad \theta_o = \frac{\sum_{r=1}^{s} y_{ro} u_{ro}}{\sum_{i=1}^{m} x_{io} v_{io}}$$

$$\text{s.t.} \quad \frac{\sum_{r=1}^{s} y_{rj} u_{ro}}{\sum_{i=1}^{m} x_{ij} v_{io}} \le 1, \quad j = 1, \cdots, n,$$

$$u_{ro} \ge \varepsilon > 0, \quad r = 1, \cdots, s,$$

$$v_{io} \ge \varepsilon > 0, \quad i = 1, \cdots, m.$$

It is claimed that the object DMU_o is efficient with the measure $\theta_o^* = 1$. We define $E = \{ j | \theta_j^* = 1, j = 1, \ldots, n\}$ to represent the set of *eDMU*s. DEA emphasizes the best individual performance of the object DMU_o. However, we always find situations where there are some *DMU*s possessing the identical individual performance, for example, those on the DEA efficient frontier.

2.3 Common Weight Analysis (CWA)

Liu & Peng (2004a) develop CWA that determines a *common* set of indices' weights, $(U_1, \ldots, U_s, V_1, \ldots, V_m)$. It is assumed all the *eDMU*s are equally weighted for determining the common set of weights. The implicit efficiencies' datum is set to one. Employing the common set of weights, *eDMU*s' implicit efficiencies could be equal one or less than one. The sum of the implicit efficiencies is the maximum, or the total of the gaps to the datum is minimized.

(P2) CWA-FP

$$\min \quad \sum_{j \in E} \left(\Delta_j^O + \Delta_j^I \right)$$

$$\text{s.t.} \quad \frac{\sum_{r=1}^{s} y_{rj} U_r + \Delta_j^O}{\sum_{i=1}^{m} x_{ij} V_i - \Delta_j^I} = 1, \quad j \in E,$$

$$\Delta_j^O, \Delta_j^I \ge 0, \quad j \in E,$$

$$U_r \ge \varepsilon > 0, \quad r = 1, \cdots, s,$$

$$V_i \ge \varepsilon > 0, \quad i = 1, \cdots, m.$$

ε is a very small positive coefficient to insure the fractional equation could be converted to linear equation by multiplication.

Liu & Peng (2004b) introduce *Super Common Weight Analysis* (SCWA) that determines a common set of indices' weights. It is assumed all the *eDMU*s are equally weighted for determining the common set of weights. The implicit efficiencies' datum is set to one. Employing the common set of weights, *eDMU*s' implicit efficiencies could be equal one or more than one. The sum of the implicit efficiencies is the minimum, or the total of the gaps to datum is minimized.

(P3) SCWA-FP

$$
\begin{aligned}
\min \quad & \sum_{j\in E}\left(\Delta_j^O+\Delta_j^I\right)\\
\text{s.t.} \quad & \frac{\sum_{r=1}^{s} y_{rj}U_r-\Delta_j^O}{\sum_{i=1}^{m} x_{ij}V_i+\Delta_j^I}=1, \quad j\in E,\\
& \Delta_j^O,\Delta_j^I\geq 0, \quad j\in E,\\
& U_r\geq \varepsilon>0, \quad r=1,\cdots,s,\\
& V_i\geq \varepsilon>0, \quad i=1,\cdots,m.
\end{aligned}
$$

3 Our Methodology

3.1 The Most Compromising Set of Indices' Weights Analysis (MCWA)

This paper presents a method that determines the most compromising set of indices' weights. It is assumed all the *eDMU*s are equally weighted for the determination. The datum of implicit efficiency is set to one. Employing the set of weights, *eDMU*s' implicit efficiencies could be above, equal, or below the datum. The total of the gaps to the datum is minimized.

The approach is analog to the ordinary least-square method (OLS) of the residuals in statistical regression analysis.

Table 2: Data Set of MCWA

DMU	Input Indices				Output Indices			
	X_1	X_2	$\cdots$	X_m	Y_1	Y_2	$\cdots$	Y_s
1	x_{11}	x_{21}	$\cdots$	x_{m1}	y_{11}	y_{21}	$\cdots$	y_{s1}
2	x_{12}	$\ddots$		x_{m2}	y_{12}	$\ddots$		y_{s2}
$\vdots$	$\vdots$			$\vdots$	$\vdots$			$\vdots$
n	x_{1n}	x_{2n}	$\cdots$	x_{mn}	y_{1n}	y_{2n}	$\cdots$	y_{sn}

Table 2 depicts there are n DMUs and their performance on the indices X_1, X_2, $\cdots$, X_m and Y_1, Y_2, $\cdots$, Y_s are given and denoted as x_{ij}, y_{rj} ($i=1,\cdots,m$; $r=1,\cdots,s$; $j=1,\cdots,n$), respectively. Given an *arbitrary* set of specific weights solution ($V_i^{\#}$, $i=1,\cdots,m$; $U_r^{\#}$, $r=1,\cdots,s$), any *DMU*, say, DUM_j's coordination's on the graph are ($\Sigma_{(i=1,\ldots,m)}x_{ij}V_i^{\#}$, $\Sigma_{(r=1,\ldots,s)}y_{rj}U_r^{\#}$). Suppose there is a DUM_j holds $\Sigma_{(i=1,\ldots,m)}x_{ij}V_i^{\#}$ =

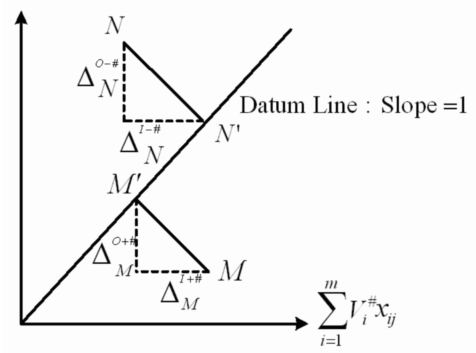

Figure 1: Gaps to the datum line in MCWA

$\Sigma_{(r=1,\ldots,s)} y_{rj} U_r^{\#}$. The line originates from the origin and pass the DUM_j is called as the *datum* line with slop equals 1, as depicted in Figure 1. The locations of DMU_M and DMU_N and their projection points on the datum line, M' and N' are identified as well. For DMU_N locates above the datum line, its projection point locates on $(\Sigma_{(i=1,\ldots,m)} x_{iN} V_i^{\#} + \Delta_N^{I-\#}, \Sigma_{(r=1,\ldots,s)} y_{rN} U_r^{\#} \bigcap \Delta_N^{O-\#})$. On the other hand, for DMU_M locates below the datum line, its projection point locates on $(\Sigma_{(i=1,\ldots,m)} x_{iM} V_i^{\#} \bigcap \Delta_M^{I+\#}$, $\Sigma_{(r=1,\ldots,s)} y_{rM} U_r^{\#} + \Delta_M^{I+\#})$. In the viewpoint of L_1-norm, its total gap is $(\Delta_M^{O+\#} + \Delta_M^{I+\#})$. Similarly, DMU_N that locates above the datum line, its total gap $(\Delta_N^{O-\#} + \Delta_N^{I-\#})$. For any *DMU*, say DMU_j, despite it locates below or above the datum line, its projection point location could be expressed as $(\Sigma_{(i=1,\ldots,m)} x_{ij} V_i^{\#} \bigcap \Delta_j^{I\#}, \Sigma_{(r=1,\ldots,s)} y_{rj} U_r^{\#} + \Delta_j^{O\#}$), where $\Delta_j^{I\#}$ and $\Delta_j^{O\#}$ are free in sign.

Model (P4) is formulated to search an optimal datum line that would result a minimum total gaps of the n DMUs. The optimal solution (V_i^* , $i = 1,\cdots,m$; U_r^*, $r = 1,\cdots,s$) is the most compromised weights.

(P4) MCWA-FP 1

$$\begin{aligned}
\min \quad & \sum_{j\in E} \left(|\Delta_j^O| + |\Delta_j^I|\right) \\
\text{s.t.} \quad & \frac{\sum_{r=1}^{s} y_{rj} U_r + \Delta_j^O}{\sum_{i=1}^{m} x_{ij} V_i - \Delta_j^I} = 1, \quad j \in E, \\
& U_r \geq \varepsilon > 0, \quad r = 1,\cdots,s, \\
& V_i \geq \varepsilon > 0, \quad i = 1,\cdots,m, \\
& \Delta_j^O, \Delta_j^I free, \quad j \in E.
\end{aligned}$$

Here, capital letter V_i and U_r denote the most compromising weights of the i-th input and the r-th output indices, respectively. It's not the *relative* weights are defined in traditional DEA models. ε is still a positive Archimedean infinitesimal constant;

we do this to avoid a case of zero value of indices obtained by choosing the set of zero weights. E denotes the set that composed by all the *eDMU*s. If Δ_j^O and Δ_j^I are unequally weighted, the right-hand-side of the equality constraint is subject to altered. We translate the above model (P4) to the following model (P5).

(P5) MCWA-FP 2

$$\min \quad \sum_{j \in E}^{\left[\left(\Delta_j^{O+}+\Delta_j^{O-}\right)+\left(\Delta_j^{I+}+\Delta_j^{I-}\right)\right]}$$

$$\begin{aligned}
\text{s.t.} \quad & \frac{\sum_{r=1}^{s} y_{rj}U_r + \left(\Delta_j^{O+} - \Delta_j^{O-}\right)}{\sum_{i=1}^{m} x_{ij}V_i - \left(\Delta_j^{I+} - \Delta_j^{I-}\right)} = 1, \quad j \in E, \\
& U_r \geq \varepsilon > 0, \quad r = 1, \cdots, s, \\
& V_i \geq \varepsilon > 0, \quad i = 1, \cdots, m, \\
& \Delta_j^{O+}, \Delta_j^{O-}, \Delta_j^{I+}, \Delta_j^{I-} \geq 0, \quad j \in E.
\end{aligned}$$

Then we translate the above fractional model to the following linear programming model (P6).

(P6) MCWA-LP 1

$$\begin{aligned}
\min \quad & \sum_{j \in E} \left[\left(\Delta_j^{O+} + \Delta_j^{O-}\right) + \left(\Delta_j^{I+} + \Delta_j^{I-}\right)\right] \\
\text{s.t.} \quad & \sum_{r=1}^{s} y_{rj}U_r - \sum_{i=1}^{m} x_{ij}V_i + \left(\Delta_j^{O+} - \Delta_j^{O-}\right) + \left(\Delta_j^{I+} - \Delta_j^{I-}\right) = 0, \quad j \in E, \\
& U_r \geq \varepsilon > 0, \quad r = 1, \cdots, s, \\
& V_i \geq \varepsilon > 0, \quad i = 1, \cdots, m, \\
& \Delta_j^{O+}, \Delta_j^{O-}, \Delta_j^{I+}, \Delta_j^{I-} \geq 0, \quad j \in E.
\end{aligned}$$

In order to decrease the complexity of linear programming (P6), we can reduce the number of variables by combining the gaps of inputs and outputs into one integrated gap $\Delta_j^+ = \Delta_j^{O+} + \Delta_j^{I+}$, $\Delta_j^- = \Delta_j^{O-} + \Delta_j^{I-}$, respectively. Then, one can easily simplify (P6) to the following model (P7).

(P7) MCWA-LP 2

$$\begin{aligned}
\Delta^* = \min \quad & \Delta = \sum_{j \in E} \left(\Delta_j^+ + \Delta_j^-\right) \\
\text{s.t.} \quad & \sum_{r=1}^{s} y_{rj}U_r - \sum_{i=1}^{m} x_{ij}V_i + \Delta_j^+ - \Delta_j^- = 0, \quad j \in E, \\
& U_r \geq \varepsilon > 0, \quad r = 1, \cdots, s, \\
& V_i \geq \varepsilon > 0, \quad i = 1, \cdots, m, \\
& \Delta_j^+, \Delta_j^- \geq 0, \quad j \in E.
\end{aligned}$$

Then we translate the above linear programming model to the following dual model, named MCWA-DLP model (P8).

(P8) MCWA-DLP 1

$$
\begin{aligned}
\max \quad & \varepsilon\left(\sum_{r=1}^{s} P_r + \sum_{i=1}^{m} Q_i\right) \\
\text{s.t.} \quad & \sum_{j\in E} y_{rj}\pi_j + P_r = 0, \quad r = 1,\cdots,s, \\
& \sum_{j\in E} x_{ij}\pi_j - Q_i = 0, \quad i = 1,\cdots,m, \\
& \sum_{j\in E} \pi_j \le 1, \\
& -\sum_{j\in E} \pi_j \le 1, \\
& P_r \ge 0, \quad r = 1,\cdots,s, \\
& Q_i \ge 0, \quad i = 1,\cdots,m, \\
& \pi_j \text{ free}, \quad j \in E.
\end{aligned}
$$

We can combine the above constraints to obtain the following dual model (P9).

(P9) MCWA-DLP 2

$$
\begin{aligned}
\max \quad & \varepsilon\left(\sum_{r=1}^{s} P_r + \sum_{i=1}^{m} Q_i\right) \\
\text{s.t.} \quad & \sum_{j\in E} y_{rj}\pi_j + P_r = 0, \quad r = 1,\cdots,s, \\
& \sum_{j\in E} x_{ij}\pi_j - Q_i = 0, \quad i = 1,\cdots,m, \\
& -1 \le \sum_{j\in E} \pi_j \le 1, \\
& P_r \ge 0, \quad r = 1,\cdots,s, \\
& Q_i \ge 0, \quad i = 1,\cdots,m, \\
& \pi_j \text{ free}, \quad j \in E.
\end{aligned}
$$

Analyzing model (P9), we observe that the setting of parameter ε can be any arbitrary positive number. In other words, we will obtain the equivalent results no matter what the value of parameter ε is. It is convenient to set the parameter $\varepsilon = 1$. Furthermore, we can take advantage of (P9) to make some appropriate illustrations about the improvement of low-ranking *DMU*s. P_r and Q_i are separately the total difference of all *DMU*s to the datum line corresponding in the r-th output and the i-th input. Furthermore, P_r and Q_i can be partitioned as $P_r = \sum_{(j\in E)} \quad p_{rj}$ and $Q_i = \sum_{(j\in E)} \quad q_{ij}$ in (P10). p_{rj} and q_{ij} are the difference of DMU_j in r-th the output and i-th input, respectively, to the datum line.

(P10) MCWA-DLP 2

$$
\begin{aligned}
\max \quad & \sum_{j \in E} \left(\sum_{r=1}^{s} p_{rj} + \sum_{i=1}^{m} q_{ij} \right) \\
\text{s.t.} \quad & \sum_{j \in E} y_{rj} \pi_j = -\sum_{j \in E} p_{rj}, \quad r = 1, \cdots, s, \\
& \sum_{j \in E} x_{ij} \pi_j = \sum_{j \in E} q_{ij}, \quad i = 1, \cdots, m, \\
& -1 \leq \sum_{j \in E} \pi_j \leq 1, \\
& p_{rj} \geq 0, \quad r = 1, \cdots, s, j \in E, \\
& q_{ij} \geq 0, \quad i = 1, \cdots, m, j \in E, \\
& \pi_j \text{ free}, j \in E.
\end{aligned}
$$

The superscript * sign denotes the optimal solutions of the decision variable in linear programming models. The solution of (P10) could be obtained indirectly by the following theorem.

Theorem 1. *The difference p_{rj}^* and q_{ij}^* of DMU_j to the datum line corresponding to the output index r and input index i are $[P_r^*\ (\Delta_j^{+*} - \Delta_j^{-*})/\ \Delta^*]$ and $[Q_i^*\ (\Delta_j^{+*} - \Delta_j^{-*})/\ \Delta^*]$.*

Proof. We claim that

$$\sum_{r=1}^{s} U_r^* \left(y_{rj} + P_r^* \frac{\Delta_j^{+*} - \Delta_j^{-*}}{\Delta^*} \right) \Bigg/ \sum_{i=1}^{m} V_i^* \left(x_{ij} + Q_i^* \frac{\Delta_j^{+*} - \Delta_j^{-*}}{\Delta^*} \right) = 1 \tag{3}$$

It is convenient to analyze this from two parts, the numerator and denominator. We can simplify the numerator

$$\sum_{r=1}^{s} U_r^* \left(y_{rj} + P_r^* \frac{\Delta_j^{+*} - \Delta_j^{-*}}{\Delta^*} \right) \tag{4}$$

as

$$\sum_{r=1}^{s} U_r^* y_{rj} + \sum_{r=1}^{s} U_r^* P_r^* \frac{\Delta_j^{+*} - \Delta_j^{-*}}{\Delta^*} \tag{5}$$

and the denominator

$$\sum_{i=1}^{m} V_i^* \left(x_{ij} + Q_i^* \frac{\Delta_j^{+*} - \Delta_j^{-*}}{\Delta^*} \right) \tag{6}$$

as

$$\sum_{i=1}^{m} V_i^* x_{ij} + \sum_{i=1}^{m} V_i^* Q_i^* \frac{\Delta_j^{+*} - \Delta_j^{-*}}{\Delta^*} \tag{7}$$

Since

$$\Delta^* = \sum_{r=1}^{s} U_r^* P_r^* + \sum_{i=1}^{m} V_i^* Q_i^*, \tag{8}$$

$$\sum_{r=1}^{s} U_r^* y_{rj} + \sum_{r=1}^{s} U_r^* P_r^* \frac{\Delta_j^{+*} - \Delta_j^{-*}}{\Delta^*} - \sum_{i=1}^{m} V_i^* x_{ij} + \sum_{i=1}^{m} V_i^* Q_i^* \frac{\Delta_j^{+*} - \Delta_j^{-*}}{\Delta^*} \tag{9}$$

$$= \sum_{r=1}^{s} U_r^* y_{rj} - \sum_{i=1}^{m} V_i^* x_{ij} + \frac{\Delta_j^{+*} - \Delta_j^{-*}}{\Delta^*} \left(\sum_{r=1}^{s} U_r^* P_r^* + \sum_{i=1}^{m} V_i^* Q_i^* \right) \tag{10}$$

$$= \sum_{r=1}^{s} U_r^* y_{rj} - \sum_{i=1}^{m} V_i^* x_{ij} + \Delta_j^{+*} - \Delta_j^{-*} = 0 \tag{11}$$

This follows the constraint condition of (P7). Hence,

$$\sum_{r=1}^{s} U_r^* \left(y_{rj} + P_r^* \frac{\Delta_j^{+*} - \Delta_j^{-*}}{\Delta^*} \right) \Bigg/ \sum_{i=1}^{m} V_i^* \left(x_{ij} + Q_i^* \frac{\Delta_j^{+*} - \Delta_j^{-*}}{\Delta^*} \right) = 1 \tag{12}$$

□

3.2 Ranking Rules of MCWA

In this section, we introduce the MCWA ranking rules. First, the MCWA-efficiency score of DMU_j is defined as the following equation.

$$\theta_j^{M*} = \frac{\sum_{r=1}^{s} U_r^* y_{rj}}{\sum_{i=1}^{m} V_i^* x_{ij}}, \quad j \in E \tag{13}$$

Before defining the ranking, we first distinguish the DMUs as three separate groups.

Definition 1. DMU_j locates on the datum line iff $\Delta_j^{+*} + \Delta_j^{-*} = 0$ or $\theta_j^{M*} = 1$. And DMU_j locates above the datum line iff $\theta_j^{M*} > 1$. Otherwise, DMU_j locates below the datum line.

It is necessary to establish this rule in order to distinguish DMUs. We can finish the ranking by the following principles.

Definition 2. The performance of DMU_{j1} is better than DMU_{j2} iff the efficiency score $\theta_{j1}^{M*} > \theta_{j2}^{M*}$.

Definition 3. If $\theta_{j1}^{M*} = \theta_{j2}^{M*} < 1$, DMU_{j1} outperforms DMU_{j2} iff $\Delta_{j1}^{+*} + \Delta_{j1}^{-*} < \Delta_{j2}^{+*} + \Delta_{j2}^{-*}$. If $\theta_{j1}^{M*} = \theta_{j2}^{M*} > 1$, DMU_{j1} outperforms DMU_{j2} iff $\Delta_{j1}^{+*} + \Delta_{j1}^{-*} > \Delta_{j2}^{+*} + \Delta_{j2}^{-*}$.

It is not necessary to presume the imaginary goal as our datum line. In fact, we can take the DMU which locates on the datum line as the standard. Therefore, we have to ensure that there is at least one DMU locates on the datum line.

Theorem 2. *There is at least one DMU locates on the datum line.*

Proof. It is claimed that we get the optimal criteria value with variables common weight U_r^*, V_i^* and $\Delta_j^{+*}, \Delta_j^{-*}$ where $\Delta_j^{+*}, \Delta_j^{-*} > 0$ for all $j \in E$ in P(7). Then, if none of the *DMU*s is on the datum line, we have two cases

Case 1: We can set the number k_j such that

$$\sum_{r=1}^{s} k_j y_{rj} U_r^* / \sum_{i=1}^{m} x_{ij} V_i^* = 1, \quad \forall j \in \text{those } DMU\text{s under datum line and } k_j > 1 \tag{14}$$

Let K= min $\{k_j\ ,\ j \in$ those *DMU*s under datum line$\}$.

The new set of common weights KU_r^* and invariant V_i^* will decrease the Δ_j^{+*} of the *DMU*s under the datum line and increase the Δ_j^{-*} of the *DMU*s above the datum line.

Case 2: We can set the number k_j such that

$$\sum_{r=1}^{s} k_j y_{rj} U_r^* / \sum_{i=1}^{m} x_{ij} V_i^* = 1, \quad \forall j \in \text{those } DMU\text{s above datum line and } k_j < 1 \tag{15}$$

Let K= max $\{k_j\ ,\ j \in$ those *DMU*s above datum line$\}$.

The new variables KU_r^* and invariant V_i^* will increase the Δ_j^{+*} of the *DMU*s under the datum line and decrease the Δ_j^{-*} of the *DMU*s above the datum line.

Either case 1 or case 2 will result in smaller criteria values and contradicts the fact that the current criteria value has been optimized. We can get another set of common weights KU_r^* and V_i^* with at least one $\Delta_j^{+*} + \Delta_j^{-*} = 0$. Hence, there is at least one *DMU* locates on the datum line. □

Definition 4. If $\theta_{j1}^{M*} = \theta_{j2}^{M*} = 1$, i.e. they are both MCWA-datum, DMU_{j1} outperforms DMU_{j2} iff the shadow price $\pi_{j1}^* < \pi_{j2}^*$.

The variable π_j in (P9) is the *shadow price*, Dantzig et al. (1997), corresponding to each constraint of (P7). The following equation shows that if the right-hand side of the j-th constraint increases 1 unit.

$$\sum_{r=1}^{s} y_{rj} U_r - \sum_{i=1}^{m} x_{ij} V_i + \Delta_j^+ - \Delta_j^- = 0 + 1 \tag{16}$$

By the definition of shadow price, the criteria, following expression, in (P7) will get the variation π_j.

$$\left(\sum_{j \in E} \left(\Delta_j^{+*} + \Delta_j^{-*} \right) \right) + \pi_j^* (0+1) \tag{17}$$

In this case, π_j^* means the impact to total gaps resulted by that the j-th MCWA-datum *DMU*s desire to raise its efficiency. Therefore we can obtain the different changed values from each MCWA-datum *DMU*s. By the way, if we have multiple MCWA-datum *DMU*s, π_j^* will appeal the good information which one has the most influence on the total gaps to assess them. By the above definitions, the *eDMU*s could be ranked.

3.3 Application of MCWATwo-Phased Method

(P7) usually results alternative solutions. The alternative sets of index weights lead to different ranking. In order to select a set of appropriate unique set of weights, the two-phased method is necessary.

Phase 1

Solve (P11) and obtain the optimal value Δ^*.

(P11)

$$\begin{aligned}
\Delta^* = \min \quad & \Delta = \sum_{j\in E}(\Delta_j^+ + \Delta_j^-) \\
\text{s.t.} \quad & \sum_{r=1}^{s} y_{rj}U_r - \sum_{i=1}^{m} x_{ij}V_i + \Delta_j^+ - \Delta_j^- = 0, \quad j \in E, \\
& U_r \geq \varepsilon, \quad r = 1,\cdots,s, \\
& V_i \geq \varepsilon, \quad i = 1,\cdots,m, \\
& \Delta_j^+, \Delta_j^- \geq 0, \quad j \in E.
\end{aligned}$$

In model (P9), the value of ε would not effect the decision variables. We set ε in the range of 10^{-6} to 10^{+6} in our numerical examples and obtained same results.

Phase 2

Solve the following linear programming (P12) to obtain the set of unique common weights.

(P12)

$$\begin{aligned}
\min \quad & \sum_{r=1}^{s} U_r - \sum_{i=1}^{m} V_i \\
\text{s.t.} \quad & \sum_{r=1}^{s} y_{rj}U_r - \sum_{i=1}^{m} x_{ij}V_i + \Delta_j^+ - \Delta_j^- = 0, \quad j \in E, \\
& \sum_{j\in E}(\Delta_j^+ + \Delta_j^-) = \Delta^* \\
& U_r \geq \varepsilon, \quad r = 1,\cdots,s, \\
& V_i \geq \varepsilon, \quad i = 1,\cdots,m, \\
& \Delta_j^+, \Delta_j^- \geq 0, \quad j \in E.
\end{aligned}$$

Now, a brief explanation of phase 2 follows. Obata and Ishii (2003) proposed that, regarding output data, it is better to adopt the smaller weights vector to obtain the product while the same product exists. This means that preference of the same product resulted from the data rather than from the weights. Similarly, it is better to use the larger weights vector for input data. This paper decides the scale of weights vector from the viewpoint of L_1-norm. We cannot prove that the weights vector is absolutely unique. But we have reduced the probability of alternative optimal weights as much as possible.

3.4 The Relation Between MCWA and Multiple Regression

For the special case, *DMU*s are assessed with two inputs and one output, the constraint in (P7) becomes:

$$y_jU_1 - x_{1j}V_1 - x_{2j}V_2 + \Delta_j^+ - \Delta_j^- = 0, \quad j \in E \tag{18}$$

It is transformed as:

$$y_j = x_{1j}\frac{V_1}{U_1} + x_{2j}\frac{V_2}{U_1} - \frac{\Delta_j^+}{U_1} + \frac{\Delta_j^-}{U_1} = 0, \quad j \in E \tag{19}$$

Let $V_1/U_1 = \alpha_1$, $V_2/U_1 = \alpha_2$, and $-\Delta_j^+/U_1 + \Delta_j^-/U_1 = e_j^M$, the constraint is rewrote as:

$$y_j = \alpha_1 x_{1j} + \alpha_2 x_{2j} \tag{20}$$

The equation has same form of multiple regression equation (2).

4 Numerical Example

4.1 Data and Results

We use the Solver in Microsoft EXCEL to carry out the calculation of linear programming models. Firstly, we use CCR or BCC model to assess *DMU*s into efficient and inefficient ones. Secondly, we use MCWA model to determine the most compromising indices' weights for those *eDMU*s. Finally, we rank those *eDMU*s.

Consider following simple illustrative problem consisting of five *DMU*s with two inputs and one output. The data set is exhibited in the next table. Note *DMU* E has extra large scale size.

Table 3: Example Data 1

DMU_j	x_{1j}	x_{2j}	y_j
A	5	12	2
B	18	16	5
C	15	9	3
D	10	12	3
E	50	59	15

Employ BCC-DEA model, these five *DMU*s are BCC-efficient. According to MCWA, DMU_E's efficiency is equal to 1. In the following table, one observes that DMU_E is the datum. The set of the most compromising weights is strongly affected by the *DMU*s with large scale. But they are not guaranteed to be a datum.

The ranks of other *DMU*s are listed in the following table. Furthermore, the differences of each *DMU* to the datum line to its reference in all indices are listed. Therefore, we have to note that the *DMU* ranked first is not sure the best reference of all *DMU*s.

Table 4: The Ranking to the Performance of BCC-*eDMU*s

Rank	DMU_j	θ_j^{M*}	Δ_j^{+*}	Δ_j^{-*}	Shadow price π_j^*	p_{1j}^*	q_{1j}^*	q_{2j}^*
1	B	1.069	0	2.333	-0.056	0	0.648	1.685
2	E	1	0	0	0.056	0	0	0
3	D	0.991	0.2	0	0.056	0	0.056	0.144
4	C	0.908	2.2	0	0.056	0	0.611	1.589
5	A	0.855	2.467	0	0.056	0	0.685	1.781

4.2 Comparison between MCWA and Regression Analysis

In the following two tables, we can find that the results of MCWA and Regression Analysis are similar, and the difference might result from diverse slopes between datum line and regression line.

Table 5: Summary of MCWA and Multiple Regression Results

DMU_j	MCWA			Regression
	Δ_j^{+*}/U_1^*	Δ_j^{-*}/U_1^*	e_j^M	e_j^R
A	0.339	0	-0.339	-0.117
B	0	0.321	0.321	0.356
C	0.303	0	-0.303	-0.294
D	0.028	0	-0.028	0.105
E	0	0	0	-0.049
sum	$\sum_j \lvert e_j^M\rvert = 0.991$			$\sum_j \lvert e_j^R\rvert = 0.921$

Table 6: The Parameters of MCWA and Multiple Regression

MCWA		Regression	
$\alpha_1 = V_1^*/U_1^*$	0.138	$\hat{\beta}_1$	0.156
$\alpha_2 = V_2^*/U_1^*$	0.138	$\hat{\beta}_2$	0.126

The MCWA equation is

$$y_j = \alpha_1 x_{1j} + \alpha_2 x_{2j} + e_j^M = 0.138x_{1j} + 0.138x_{2j} + e_j^M.$$

The regression equation is

$$y_j = \hat{\beta}_1 x_{1j} + \hat{\beta}_2 x_{2j} + e_j^R = 0.156x_{1j} + 0.126x_{2j} + e_j^R.$$

4.3 Comparison of CWA, SCWA and MCWA

Consider following illustrative problem consisting of fifteen *DMU*s with two inputs and two outputs. The data set is exhibited in the following table.

Table 7: Example Data 2

DMU_j	x_{1j}	x_{2j}	y_{1j}	y_{2j}
A	8	3	20	45
B	33	2	33	52
C	6	2	2	31
D	25	4	63	52
E	2	36	52	1
F	6	38	30	49
G	9	3	40	3
H	35	12	88	44
I	15	133	223	32
J	100	61	15	333
K	33	11	66	99
L	553	111	55	321
M	22	13	95	22
N	51	5	94	11
O	4	7	9	3

In the following table, we obtain different ranking for CWA, SCWA and MCWA. But we can find that *DMU* A outperforms the others and DMU_L possesses the worst performance among the three methods.

5 Conclusion

This paper proposes a model to discriminate the efficient *DMU*s in DEA models. This new ranking method based on the DEA framework helps the decision maker (DM) in the task of assessment. We provide the ranking rule as a convenient guide for DM to judge which *DMU* the best is. In the cases of identical individual performance, we rank the contributions of each *DMU* to the total gaps. Simultaneously, we provide the difference of each input and output index to the datum line for the *DMU*s not on the datum line.

There are a number of future research issues that remain to be addressed. One of them is that how to extend the full ranking to the DEA-*iDMU*s. There is no guarantee that MCWA will result in *eDMU*s earning good ranking, simply because MCWA is not emphasized in the individual performance. Hence, there remains the critical issue of how to further balance both the optimization of individual performance and overall performance at the same time.

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Table 8: Summary of CWA, SCWA and MCWA Results

Rank	CWA			SCWA			MCWA			
	DMU_j	θ_j^{C*}	Δ_j^{+*}	DMU_j	θ_j^{S*}	Δ_j^{-*}	DMU_j	θ_j^{M*}	Δ_j^{+*}	Δ_j^{-*}
1	A	1	0	A	9.1	99.97	A	4.85	0	51.59
2	I	1	0	C	7.38	56.7	D	3.57	0	82.78
3	E	0.99	0.09	D	5.51	138.89	C	3.43	0	23.39
4	F	0.92	7.11	K	5.5	220.18	K	3.12	0	112.15
5	C	0.68	15.49	B	3.89	103.78	G	2.98	0	28.59
6	M	0.64	66.71	J	3.71	509.91	M	2.57	0	71.54
7	K	0.62	101.69	G	3.46	32.82	H	2.33	0	75.35
8	G	0.59	29.73	M	3.44	99.33	B	2.32	0	48.39
9	D	0.58	82.51	H	3.41	125.91	N	1.75	0	44.98
10	H	0.47	151.2	F	2.14	69.57	J	1.66	0	137.93
11	J	0.42	489.06	N	2	58.34	F	1.06	0	4.43
12	B	0.33	172.28	I	1.39	81.34	I	1	0	0
13	O	0.31	26.25	O	1.07	1.03	E	0.79	13.96	0
14	N	0.26	294.61	E	1	0	O	0.72	4.63	0
15	L	0.09	4016.4	L	1	0	L	0.5	377.31	0
	$\Delta^* = 5453.04$			$\Delta^* = 1597.77$			$\Delta^* = 1077$			
Weights	$V_1^*= 7.73$ $V_2^*= 1.05$ $U_1^*= 1$ $U_2^*= 1$			$V_1^*= 1$ $V_2^*= 1.45$ $U_1^*= 1$ $U_2^*= 2.05$			$V_1^*= 1$ $V_2^*= 1.8$ $U_1^*= 1$ $U_2^*= 1$			

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