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Some Results on the Classification for f-colored Graphs*

Jiguo Yu[1,†] Lihua Han[1] Guizhen Liu[2]

[1] School of Computer Science, Qufu Normal University, Rizhao, Shandong 276826, China
[2] School of Mathematics and system science, Shandong University, Jinan, Shandong 250100, China

Abstract f-colorings have applications in scheduling problems. An f-coloring of a graph G is a coloring of edges of $E(G)$ such that each color appears at each vertex $v \in V(G)$ at most $f(v)$ times. The minimum number of colors needed to f-color G is called the f-chromatic index of G, and denoted by $\chi'_f(G)$. Any graph G has f-chromatic index equal to $\Delta_f(G)$ or $\Delta_f(G)+1$, where $\Delta_f(G) = \max\limits_{v\in V}\{\lceil \frac{d(v)}{f(v)} \rceil\}$. If $\chi'_f(G) = \Delta_f(G)$, then G is of C_f 1; otherwise G is of C_f 2. The f-core of G is the subgraph of G induced by the vertices of $V_0^* = \{v : \Delta_f(G) = \frac{d(v)}{f(v)}, v \in V\}$. In this paper, some conditions for the classification on f-coloring are given.

Keywords edge-coloring; f-coloring; classification of graphs

1 Introduction

Our terminology and notation in this paper are standard. Readers are referred to [1] for undefined terms. Throughout this paper, the graph refers to a simple graph. A multigraph may have multiple edges but no loops. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For each $v \in V(G)$, $d_G(v)$ denotes the degree of v, and $N_G(v)$ denotes the vertex set adjacent to v. $\Delta(G)$ is the maximum degree of G, and $\delta(G)$ is the minimum degree of G. Let f be a positive integer-valued function defined on $V(G)$. A graph G is called a fan-graph, if it can be obtained from a path $P_k = v_1v_2...v_k (k \geq 2)$ by adding a new vertex w and joining w to all the vertices on the path. w is called the core. If a circuit have k edges, then the circle is called k-circuit. A wheel G is a graph obtained from a k-circuit by adding a new vertex w and then joining this new vertex to all the vertices on the circuit. A wheel G is an even wheel if k is even and an odd wheel otherwise. w is also called the core. A graph G is called series-parallel graph if G has no subgraph homeomorphic to K_4.

The edge-coloring problem was posed in 1880 in relation with the well-known four-color conjecture. The four-color conjecture is that every map could be colored

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†Corresponding author. E-mail: jiguoyu@sina.com, jgyu@qrnu.edu.cn

with four colors so that any neighboring countries have different colors. It took more than 100 years to prove the conjecture affirmatively in 1976 with the help of computers since it was posed in 1852. In the proper edge-coloring, each vertex has at most one edge colored with a given color. The minimum number of colors needed to color the edges of G in such a way that no two adjacent edges are assigned the same color is called the chromatic index, denoted by $\chi^{'}(G)$. Hakimi and Kariv [1] generalized the proper edge-coloring and obtained many interesting results.

An f-coloring of G is a coloring of edges such that each vertex v has at most $f(v)$ edges colored with the same color. The minimum number of colors needed to f-color G is called the f-chromatic index of G, and denoted by $\chi^{'}_f(G)$. If $f(v) = 1$ for all $v \in V$, the f-coloring problem is reduced to the proper edge-coloring problem.

f-colorings have applications in scheduling problems such as the file transfer problem in a computer network [2-4, 6]. The file transfer problem on computer networks is molded as follows. Each computer v has a limited number $f(v)$ of communication ports. For each pair of computer there are a number of files which are transferred between the pair of computers. In such a situation the problem is how to schedule the file transfers so as to minimize the total time for the overall transfer process. The file transfer problem in which each file has the same length is formulated as an f-coloring problem for a graph as follows. Vertices of the graph correspond to nodes of the network, and edges correspond to files to be transferred between the endpoints. Such a graph G describes the file transfer demands. Assume that each computer v has $f(v)$ communication ports, and transferring any file take an equal amount of time. Under these assumptions, the schedule to minimize the total time for overall transfer process corresponds to an f-coloring of G with the minimum number of colors. Note that the edges colored with the same color correspond to files that can be transferred simultaneously.

Since the proper edge-coloring problem is NP-complete [5], the f-coloring problem which asks us to find $\chi^{'}_f(G)$ of a given multigraph G is also NP-complete in general. In the proper edge-coloring, one of the most celebrated results is that $\chi^{'}(G) = \Delta(G)$ or $\Delta(G)+1$ for any graph G, which is due to Vizing [5]. This result naturally partitions all graphs into two classes, and we say that G is class 1 if $\chi^{'}(G) = \Delta(G)$, and class 2 otherwise.

Let

$$\Delta_f(G) = \max_{v\in V}\{\lceil \frac{d(v)}{f(v)} \rceil\}.$$

in which $\lceil \frac{d(v)}{f(v)} \rceil$ is the smallest integer not smaller than $\frac{d(v)}{f(v)}$. It is easy to verify that $\chi^{'}_f(G) \geq \Delta_f(G)$. The multiplicity $\mu(u,v)$ of a pair of u and v of distinct vertices is the number of edges joining u and v. Let $\mu(v) = \max_{u\in V}\{\mu(v,u)\}$. The following lemma was given by Hakimi and Kariv [4].

Theorem 1. *Let G be a multigraph, Then*

$$\Delta_f(G) \leq \chi^{'}_f(G) \leq \max_{v\in V}\{\lceil \frac{d(v)+\mu(v)}{f(v)} \rceil\}.$$

When G is a graph, we have $\mu(v) \leq 1$ for each $v \in V$. Therefore the following lemma holds.

Theorem 2. *Let G be a graph. Then*

$$\Delta_f(G) \leq \chi_f'(G) \leq \max_{v \in V}\{\lceil \frac{d(v)+1}{f(v)} \rceil\} \leq \Delta_f(G)+1.$$

>From the above lemma we can see that the f-chromatic index of any graph G be $\Delta_f(G)$ or $\Delta_f(G)+1$. This immediately gives us a simple way of classifying graphs into two classes according to their f-chromatic indices. More precisely, we say that G is of C_f 1 if $\chi_f'(G) = \Delta_f(G)$; and that G is of C_f 2 if $\chi_f'(G) = \Delta_f(G)+1$.

Hakimi and Kariv [1] obtained the f-chromatic indices of bipartite graphs and graphs with $f(v)$ being even for all $v \in V$. Zhang and Liu studied the classification of regular graphs and complete graphs on f-colorings[9−11].

Let

$$V^* = \{v | \Delta_f = \lceil \frac{d(v)}{f(v)} \rceil, v \in V\}.$$

and

$$V_0^* = \{v | \Delta_f = \frac{d(v)}{f(v)}, v \in V\}.$$

Theorem 3. [9] *Let G be a graph and let G_0^* be the subgraph of G induced by the vertices of V_0^*. Then G is of C_f 1 if G_0^* is a forest.*

Theorem 4. [10] *Let G be a graph. Let $f(v)$ and V^* be as defined earlier. If $f(v^*) \nmid d(v^*)$ for all $v^* \in V^*$, then G is of C_f 1.*

Theorem 5. [8] *Let G be a series-parallel graph with $\delta \geq 2$. Then at least one of the following case holds:*

(1) There exists an edge $e = uv$ such that $d(u)+d(v) \leq 5$;

(2)There exist two disjoint vertices u and v of degree 2 which have a common neighbor w of degree 4 such that $N(u) \backslash \{w\} = N(v) \setminus \{w\} \subset N(w)$;

(3)There exist two disjoint vertices u and v of degree 2 which have a common neighbor w of degree 4 such that $N(w) \setminus \{u,v\} = (N(u) \cup N(v)) \setminus \{w\} = \{x,y\}$;

(4)There exist three vertices u, v and w of degree 2 such that $N(u) = N(v)$ and $N(u) \cap N(w) \neq \phi$.

Theorem 6. [4] *Let G be a graph and $f(v)$ be even for all $v \in V$. Then $\chi_f'(G) = \Delta_f(G)$.*

In the following section, we will consider the classification of fan graphs, wheels and series-parallel graphs.

2 Main results and proofs

Before discussing the classification of graphs, we need some preliminary knowledge from [9]. We denote by C the set of $\Delta_f(G)$ colors used to f-color a graph G.

An edge colored with color $c \in C$ is called a c-edge. Denoted by $d(v,c)$ the number of c-edge of G incident with the vertex v, and define $m(v,c) = f(v) - d(v,c)$. Define $M(v) = \{c : m(v,c) \geq 1, c \in C\}$.

Theorem 7. *Let G be a fan-graph with the core w and the path $P_n = v_1v_2...v_n (n \geq 2)$. G is of C_f 1 but $n = 2$ and $f(v) = 1$ for all $v \in V$.*

Proof. If $n = 2$ and $f(v) = 1$ for all $v \in V$, then G is an odd cycle and the f-coloring reduced to proper edge-coloring. Obviously, G is of class two and G is of C_f 2. If $n = 2$ and there exits at least one vertex $v \in V(G)$ such that $f(v) \geq 2$, then we have $\Delta_f(G) = 1$ or $\Delta_f(G) = 2$. It is easy to verify that G is of C_f 1. In the following, we suppose that $n \geq 3$. Four cases need to be discussed.

Case 1. $\Delta_f(G) = 1$.

In this case, we have $d(v) \leq f(v)$ for all $v \in V$. Obviously, we can f-color graph G with one color and thus G is of C_f 1.

Case 2. $\Delta_f(G) = 2$.

In this case, if $V_0^* \neq \phi$, then G_0^* is a forest since $V_0^* \subseteq \{v_1, v_n, w\}$ by the definition of V_0^*. Thus G is of C_f 1 by Theorem 3. If $V_0^* = \phi$, then G is of C_f 1 by Theorem 4.

Case 3. $\Delta_f(G) = 3$.

In this case, it suffice to proof G is C_f 1 when $f(v_i) = 1 (i = 1, 2, ..., n)$ and $\frac{d(w)}{f(w)} = 3$.

Draw the path P_n horizontally and draw the core 2 under the path P_n. Join w to each vertex of P_n by a straight line. f-color, sequentially, the edges incident to w from left to right using the colors 1, 2 and 3. f-color , sequentially, the edges on the path from left to right using the colors 3, 1, and 2. We obtain a desired f-coloring of G with $\Delta_f(G) = 3$ colors. Hence G is of C_f 1.

Case 4. $\Delta_f(G) \geq 4$.

In this case, obviously, $V_0^* \subseteq \{w\}$. If $V_0^* = \{w\}$ then G_0^* is a forest and G is of C_f 1 by Theorem 3. If $V_0^* = \phi$, then G is of C_f 1 by Theorem 4.

The theorem is proved. □

By Theorem 7, the following corollary holds.

Corollary 8. *Let G be a fan-graph with the core w and the path $P_n = v_1v_2...v_n (n \geq 2)$. If $n \geq 3$, then G is of class one.*

Theorem 9. *Let G be a wheel of order $n+1$ with the core w and the cycle $C_n = v_1v_2 \cdots v_nv_1$. If $d(w) \neq 3r + 2 (r = 1, 2, ...)$ when $\Delta_f(G) = 3$, then G is of C_f 1.*

Proof. To prove that the theorem, for cases need to be considered.

Case 1. $\Delta_f(G) = 1$.

In this case, $d(v) \leq f(v)$ for all $v \in V$. Obviously, we can f-color graph G with one color and thus G is of C_f 1.

Case 2. $\Delta_f(G) = 2$.

In this case, we have $V_0^* \subseteq \{w\}$ by the definition of V_0^*. Then G_0^* is a forest if $V_0^* = \{w\}$, and G is of C_f 1 by Theorem 3. If $V_0^* = \phi$, G is of C_f 1, by Theorem 4.

Case 3. $\Delta_f(G) = 3$.

In this case, G is of C_f 1 if and only if $\chi'_f(G) = 3$.

Subcase 3.1. $d(w) = 3r (r \geq 1)$.

We draw the circle $C_n = v_1v_2v_3...v_nv_1$ in a clockwise direction. Starting from v_1v_2, f-color the edges on the circle with the color 1, 2, and 3 alternately. Then f-color the edges $wv_i (i = 1,2,...,n)$ with the color 2, 3, and 1 alternately. Thus, a desired f-coloring of G is obtained.

Subcase 3.2. $d(w) = 3r+1 (r \geq 1)$.

In this case, obviously, $\frac{d(w)}{f(w)} \leq 3 = \Delta_f(G)$. We say that $w \notin V_0^*$. Otherwise, $w \in V_0^*$, we have $\frac{d(w)}{f(w)} = 3$, that is $f(w) = r + \frac{1}{3}$. But $f(w)$ is an integer, a contradiction. Thus, $V_0^* \subseteq V(C_n)$. If $V_0^* \subset V(C_n)$, then G_0^* is a forest, and G is of C_f 1 by Theorem 3. If $V_0^* = V(C_n)$, we give an f-coloring of G. Note that $\Delta_f(G) = 3$, $d(v_i) = 3$ and $v_i \in V_0^*$ for all $i \in \{1,2,\cdots,n\}$, we have $f(v_i) = 1$ for all $i \in \{1,2,\cdots,n\}$. Since $\frac{d(w)}{f(w)} < 3$, we have $\lceil \frac{d(w)}{f(w)} \rceil = 3$ or $\lceil \frac{d(w)}{f(w)} \rceil < 3$. It is easy to see that it suffice to prove that $\chi'_f(G) = 3$ when $\lceil \frac{d(w)}{f(w)} \rceil = 3$. Since $\frac{d(w)}{f(w)} < 3$ and $d(w) = 3r+1$, we have $f(w) \geq r + \frac{1}{3}$. Thus $f(w) \geq r+1$ since $f(w)$ is an integer. Now, we give an f-coloring with $\chi'_f(G) = 3$ when $f(w) = r+1$ and $f(v_i) = 1$ for all $i \in \{1,2,\cdots,n\}$. Draw the circle $C_n = v_1v_2v_3...v_nv_1$ in a clockwise direction. Starting from v_1v_2, f-color the edges on the circle but v_nv_1 with the color 1, 2, and 3, alternately. Then f-color the edges $wv_i (i = 2,3,...,n-1)$ with the color 3, 1, and 2, alternately. Finally, f-color the edge v_nv_1 with the color 2. Now there are only two uncolored edges wv_n and wv_1. f-color the edge wv_n with one of the colors $c \in M(v_n)$ and f-color the edge wv_1 with one of the colors $c \in M(v_1)$. A desired f-coloring of G is obtained.

Case 4. If $\Delta_f(G) \geq 4$, then $V_0^* \subseteq \{w\}$. If $V_0^* = \{w\}$, then G_0^* is a forest and G is C_f 1 by Theorem 3. If $V_0^* = \phi$, then G is also of C_f 1 by Theorem 4.

This completes the proof of the theorem. □

Remark 1. we conjecture that G is of C_f 2 if $d(w) = 3r+2$ in Theorem 9, but we can not prove it now.

Theorem 10. *If G is a 2-connected series-parallel graph and $f(v) \geq 2$ for all $v \in V(G)$, then G is of C_f 1.*

Proof. If $\Delta = 2$, obviously, the theorem holds. It suffices to prove that the theorem holds for $\Delta \geq 3$. We proceed by induction on both the number of vertices $P(G)$ and the maximum degree $\Delta(G)$. If $\Delta(G) = 3$, we prove that G is of C_f 1. If $f(v) = 2$ for all $v \in V(G)$, then G is of C_f 1 by Theorem 4. If there exists a vertex v with $f(v) \geq 3$, then $\lceil \frac{d(v)}{f(v)} \rceil = 1$ since $\delta \geq 2$ and $\Delta = 3$, which can not change the coloring of G. Thus G is also of C_f 1. If $f(v) \geq 3$ for all $v \in V(G)$, it is easy to see $\Delta_f(G) = 1$. Thus G is of C_f 1. We may assume the theorem holds on $\Delta(G) < \Delta (\Delta \geq 4)$(the first hypothesis). In the case $\Delta(G) = \Delta$, we may proceed by induction on the number of vertices $P(G)$. It is easy to see that the theorem holds on $P(G) = 3$, since G is an 3-cycle, $\Delta_f(G) = 1$ since $f(v) \geq 2$ for all $v \in V(G)$. We may assume the theorem holds for $P(G) < P(P \geq 4)$ (the second hypothesis). We consider five cases according to Theorem 5.

Case 1. There exists an edge $e = uv$ such that $d(u) + d(v) = 4$.

Let $N(u) = \{v,x\}$ and $N(v) = \{u,y\}$. Obviously $x \neq y$, otherwise $x \equiv y$ is a cut-vertex, which contradicts the fact that G is 2-connected. Let $G^* = G \setminus \{u\} \cup \{vx\}$. If $\Delta(G^*) < \Delta$, then G^* is of C_f 1, by the first hypothesis; If $\Delta(G^*) = \Delta$, G^* is also of C_f 1, by the second hypothesis. Let σ^* be an f-coloring of G^*. Consider an f-coloring σ of G such that for all $e \in E(G) \setminus \{ux,uv\}$, we have $\sigma^*(e) = \sigma(e)$. In f-coloring σ of G, note that $m(v) > 0$, $m(x) > 0$ and $f(u) \geq 2$, we can obtain an proper f-coloring of G.

Case 2. There exists an edge $e = uv$ such that $d(u) + d(v) = 5$.

We consider two subcases.

Subcase 2.1. $N(u) \cap N(v) = \phi$, $N(u) = \{v,x\}$ and $N(v) = \{u,y_1,y_2\}$.

Let $G^* = G \setminus \{u\} \cup \{vx\}$. If $\Delta(G^*) < \Delta$, then G^* is of C_f 1, by the first hypothesis; If $\Delta(G^*) = \Delta$, then G^* is of C_f 1, by the second hypothesis. Let σ^* be an f-coloring of G^*. Consider an f-coloring σ of G such that for all $e \in E(G) \setminus \{ux,uv\}$, we have $\sigma^*(e) = \sigma(e)$. In f-coloring σ of G, note that $m(v) > 0$, $m(x) > 0$ and $f(u) \geq 2$, we can obtain an proper f-coloring of G.

Case 2.2. $N(u) \cap N(v) \neq \phi$, $N(u) = \{v,x\}$ and $N(v) = \{u,x,y_1\}$.

Let $G^* = G \setminus \{u\}$. If $\Delta(G^*) < \Delta$, then G^* is of C_f 1, by the first hypothesis; if $\Delta(G^*) = \Delta$, G^* is of C_f 1, by the second hypothesis. Let σ^* be an f-coloring of G^*. Consider an f-coloring σ of G such that for all $e \in E(G) \setminus \{ux,uv\}$, we have $\sigma^*(e) = \sigma(e)$. In f-coloring σ of G, note that $m(v) > 0$, $m(x) > 0$ and $f(u) \geq 2$, we can obtain an proper f-coloring of G.

Case 3. There exist two disjoint vertices u and v of degree 2 which have a common neighbor w of degree 4 such that $N(u) \setminus \{w\} = N(v) \setminus \{w\} \subset N(w)$.

Let $N(u) \setminus \{w\} = N(v) \setminus \{w\} = \{x\}$ and let $G^* = G \setminus \{u\}$. If $\Delta(G^*) < \Delta$, G^* is of C_f 1, by the first hypothesis; If $\Delta(G^*) = \Delta$, G^* is of C_f 1 by the second hypothesis. Let σ^* be an f-coloring of G^*. Consider an f-coloring σ of G such that for all $e \in E(G) \setminus \{ux,uw\}$, we have $\sigma^*(e) = \sigma(e)$. In f-coloring σ of G, note that $m(w) > 0$, $m(x) > 0$ and $f(u) \geq 2$, we can obtain an proper f-coloring of G.

Case 4. There exist two disjoint vertices u and v of degree 2 which have a common neighbor w of degree 4 such that $N(w) \setminus \{u,v\} = (N(u) \cup N(v)) \setminus \{w\} = \{x,y\}$.

Let $G^* = G \setminus \{u\}$. If $\Delta(G^*) < \Delta$, G^* is of C_f 1, by the first hypothesis; If $\Delta(G^*) = \Delta$, G^* is of C_f 1, by the second hypothesis. Let σ^* be an f-coloring of G^*. Consider an f-coloring σ of G such that for all $e \in E(G) \setminus \{ux,uw\}$, we have $\sigma^*(e) = \sigma(e)$. In f-coloring σ of G, note that $m(w) > 0$, $m(x) > 0$ and $f(u) \geq 2$, we can obtain an proper f-coloring of G.

Case 5. There exist three disjoint vertices u, v and w of degree 2 such that $N(u) = N(v)$ and $N(u) \cap N(w) \neq \phi$.

Let $N(u) = \{x_1, x_2\}$, $x_1 \in N(w)$. If x_1 and x_2 are not adjacent, then let $G^* = G \setminus \{u\} \cup \{x_1x_2\}$, otherwise let $G^* = G \setminus \{u\}$. If $\Delta(G^*) < \Delta$, G^* is of C_f 1, by the first hypothesis; If $\Delta(G^*) = \Delta$, G^* is of C_f 1, by second hypothesis. Let σ^* be an f-coloring of G^*. Consider an f-coloring σ of G such that for all $e \in E(G) \setminus \{ux_1, ux_2\}$, we have $\sigma^*(e) = \sigma(e)$. In f-coloring σ of G, note that $m(x_1) > 0$, $m(x_2) > 0$ and $f(u) \geq 2$, we can obtain an proper f-coloring of G.

All cases complete the proof. □

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