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Robust Impulsive Synchronization of Delayed Dynamical Networks and Its Applications

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Abstract This brief paper further investigates robust impulsive synchronization of complex delayed dynamical networks. Based on impulsive control theory on delayed dynamical systems, some simple yet less conservative criteria ensuring robust impulsive synchronization of coupled delayed dynamical networks are derived analytically. Furthermore, the theoretical results are applied to a typical scale-free (SF) network composing of the representative chaotic delayed Hopfield neural network nodes, and numerical results are presented to demonstrate the effectiveness of the proposed control techniques.

Keywords Robust impulsive synchronization; coupled delayed dynamical networks; chaotic Hopfield neural network.

1 Introduction

Recent years have witnessed increasing interest in the study of complex networks from various fields of science and engineering. Networks are present ubiquitously in the real word, including biological systems, genetic chains, protein interaction graphs, social relationships and artificial and engineering architectures, etc [1, 2, 3] .In the past decade, special attention has been focused on the synchronization dynamics in large-scale complex networks composing of coupled chaotic oscillators with small-world and scale-free characters (see [4, 5, 6] and references cited therein).

In the past several years, impulsive control has been widely used to stabilize and synchronize chaotic dynamical systems due to its potential advantages over general continuous control schemes [7, 8, 14]. It has been proved, in the study of chaos synchronization, that impulsive synchronization approach is effective and robust in synchronization of chaotic dynamical systems. Moreover, the controllers used usually have a relatively simple structure. In an impulsive synchronization scheme, only the synchronization impulses are sent to the receiving systems at the impulsive instances, which can decrease the information redundancy in the transmitted signal

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and increase robustness against the disturbances. In this sense, impulsive synchronization schemes are very useful in practical application, such as in digital secure communication systems [7, 8]. Therefore, the investigation of impulsive synchronization of complex dynamical networks is important for design and applications in engineering and technology.

This paper is mainly concerned with the issues of robust impulsive synchronization of coupled delayed dynamical networks. Based on impulsive control theory on delayed dynamical systems, some simple yet less conservative criteria are derived for robust impulsive synchronization of the coupled delayed dynamical networks. Furthermore, the obtained results are applied to a typical complex dynamical networks composing of the chaotic delayed Hopfield neural network nodes with scale-free characters, and the numerical simulations also demonstrate the effectiveness and feasibility of the proposed control techniques.

2 Problem Formulations

First, we consider a dynamical network consisting of N linearly coupled identical delayed dynamical nodes, which is described by the following set of differential equations [5]:

$$\dot{x}_i(t) \quad = \quad f(t, x_i(t), x_i(t-\tau)) + \sum_{j=1}^{N} b_{ij}\Gamma x_j(t), i = 1, 2\cdots, N. \tag{1}$$

in which $x_i(t) = (x_{i1}(t), \cdots, x_{in}(t))^\top \in R^n$ are the state variables of the ith delayed dynamical node, $f : R \times R^n \times R^n \to R^n$ is continuously vector-valued function. For simplicity, we further assume that the inner connecting matrix $\Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_n)$, and the coupling matrix $B = (b_{ij})_{N\times N}$ is a symmetric irreducible matrix with zero-sum and nonnegative off-diagonal elements. This implies that zero is an eigenvalue of B with multiplicity 1 and all the other eigenvalues of B are strictly negative [4, 5].

Next we consider an isolated identical dynamical system in the model (1), which is described by the following form of n-dimensional equations with time delays [9]:

$$\dot{x}(t) = f(t, x(t), x(t-\tau)) = Ax(t) + g(t, x(t), x(t-\tau)), \tag{2}$$

in which $x(t) = (x_1(t), \cdots, x_n(t))^\top \in R^n, A \in R^{n\times n}$, and the vector-valued function $g(t, x(t), x(t-\tau)) = (g_1(t, x(t), x(t-\tau)), \cdots, g_n(t, x(t), x(t-\tau))^\top \in R^n$. Throughout this paper, we always assume that $g(t, x(t), x(t-\tau))$ satisfy uniform Lipschitz condition with respect to the time t, i.e.,

(A_1) For any $x(t) = (x_1(t), \cdots, x_n(t))^\top \in R^n, y(t) = (y_1(t), \cdots, y_n(t))^\top \in R^n$, there exist constants $k_{ij} > 0$ satisfying

$$\begin{aligned} & |g_i(t, x(t), x(t-\tau)) - g_i(t, y(t), y(t-\tau))| \\ \leq \quad & \sum_{j=1}^{n} k_{ij}\big(|x_j(t) - y_j(t)| + |x_j(t-\tau) - y_j(t-\tau)|\big), \quad i = 1, 2, \cdots, n. \end{aligned} \tag{3}$$

Remark 1. It is easy to check that the class of systems in the form of Eqs. (2)-(3) includes almost all the well-known chaotic systems with delays or without delays such as the Lorenz system, Rössler system, Chen system, Chua's circuit as well as the delayed Mackey-Glass system or delayed Ikeda equations, delayed Hopfied neural networks and delayed cellular neural networks (CNNs), and so on (see [6, 9]).

Now we consider the issues of impulsive control for robust synchronization of the delayed dynamical network (1). By adding an impulsive controller $\{t_k, I_{ik}(t, x_i(t))\}$ to the ith-dynamical node of the network (1), we have the following impulsively controlled delayed dynamical network:

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + g(t, x_i(t), x_i(t-\tau)) + \sum_{j=1}^{N} b_{ij}\Gamma x_j(t), & t \neq t_k, t \geq t_0, \\ \triangle x_i = I_{ik}(t, x_i(t)), & t = t_k, k = 1, 2, \cdots, \end{cases} \tag{4}$$

where $i = 1, 2\cdots, N,$ the time sequence $\{t_k\}_{k=1}^{+\infty}$ satisfy $t_{k-1} < t_k$ and $\lim_{k\to\infty} t_k = +\infty$, $\triangle x_i = x_i(t_k^+) - x_i(t_k^-)$ is the control law in which $x_i(t_k^+) = \lim_{t\to t_k^+} x_i(t)$ and $x_i(t_k^-) = \lim_{t\to t_k^-} x_i(t)$. Without loss of generality, we assume that $\lim_{t\to t_k^+} x_i(t) = x_i(t_k)$, which means the solution $x(t)$ is continuous from the right. The initial conditions of Eq. (2) are given by $x_i(t) = \phi_i(t) \in PC([t_0-\tau, t_0], R^n)$, where $PC([t_0-\tau, t_0], R^n)$ denotes the set of all functions of bounded variation and right-continuous on any compact subinterval of $[t_0-\tau, t_0]$. We always assume that Eq. (4) has a unique solution with respect to initial conditions. Clearly, if $I_{ik}(t, x_i(t)) = 0$, then the controlled model (4) becomes the continuous delayed dynamical network (1).

In this paper, we define the synchronization state of the controlled delayed dynamical network (4) as $s(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$, where $x_i(t)$ $(i = 1, 2\cdots, N)$ are the solutions of the continuous delayed dynamical network (1) [13]. The main objective of this paper is to design and implement an appropriate impulsive controller $\{t_k, I_{ik}(t, x_i(t))\}$ such that the states of the controlled delayed dynamical network (4) will achieve synchronization, i. e.,

$$\lim_{t\to+\infty} \parallel x_i(t) - s(t) \parallel = 0, \qquad i = 1, 2\cdots, N, \tag{5}$$

where $s(t)$ is called as the synchronization state of the controlled delayed dynamical network (4). It may be an equilibrium point, a periodic orbit, or a chaotic attractor.

3 Robust Impulsive Synchronization

Base on impulsive control theory on delayed dynamical systems, the following sufficient condition for robust impulsive synchronization of the controlled delayed dynamical network (4) is established.

Theorem 1. *Consider the controlled delayed dynamical network (4). Let the impulsive controller as*

$$u_i(t, x_i) = \sum_{k=1}^{+\infty} I_{ik}(t, x_i(t))\delta(t-t_k) = \sum_{k=1}^{+\infty} d_k\big(x_i(t_k^-) - s(t)\big)\delta(t-t_k), \tag{6}$$

where d_k is a constant called as the control gain, $\delta(t)$ is the Dirac function, and the eigenvalues of its coupling matrix B be ordered as

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots, \lambda_N. \tag{7}$$

Assume that, in addition to (A_1), the following conditions are satisfied for all $i = 1, 2, \cdots, n$ and $k \in Z^+ = \{1, 2, \cdots, \infty\}$

(A_2) There exist n positive numbers $\delta_1, \cdots, \delta_n$, and two numbers

$$p = \min_{1\leq i\leq n} \Big\{2\delta_i - \lambda_{\max}(A + A^\top) - \sum_{s=1}^{n} (2k_{is}^{2\varepsilon} + k_{si}^{2(1-\varepsilon)})\Big\}, \tag{8}$$

$$q = \max_{1\leq i\leq n} \Big\{\sum_{s=1}^{n} k_{si}^{2(1-\varepsilon)}\Big\}, \tag{9}$$

such that $p > q$ and $\gamma_i \lambda(\gamma_i) + \delta_i \leq 0$, where $\lambda_{\max}(A + A^\top)$ is the most eigenvalue of the matrix $(A + A^\top)$, and

$$\lambda(\gamma_i) = \begin{cases} \lambda_2, & \text{if } \gamma_i > 0, \\ 0, & \text{if } \gamma_i = 0, \\ \lambda_N, & \text{if } \gamma_i < 0. \end{cases} \tag{10}$$

(A_3) Let $\mu > 0$ satisfy $\mu - p + qe^{\mu\tau} \leq 0$, and

$$\theta_k = \max\big\{1, (1 + d_k)^2\big\}, \quad \theta = \sup_{k\in Z^+} \Big\{\frac{\ln\theta_k}{t_k - t_{k-1}}\Big\} \tag{11}$$

such that $\theta < \mu$. Then the controlled delayed dynamical network (4) is robustly exponentially synchronized.

Brief Proof.

Let $v_i(t) = x_i(t) - s(t)$ $(i = 1, 2, \cdots, N)$, then the error dynamical system can be rewritten as

$$\begin{cases} \dot{v}_i(t) = Av_i(t) + A\tilde{g}(t, v_i(t), v_i(t-\tau)) + \sum_{j=1}^{N} b_{ij}\Gamma v_j(t) + J, t \neq t_k, t \geq t_0, \\ v_i(t_k) = (1 + d_k)v_i(t_k^-), \qquad t = t_k, k = 1, 2, \cdots, \end{cases} \tag{12}$$

where

$$\tilde{g}(t, v_i(t), v_i(t-\tau)) = g(t, v_i(t) + s(t), v_i(t-\tau) + s(t-\tau)) - g(t, s(t), s(t-\tau))$$

and

$$J = g(t, s(t), s(t-\tau)) + \frac{1}{N}\sum_{k=1}^{N}(t, x_k(t), x_k(t-\tau)). \tag{13}$$

Let us construct a Lyapunov function

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} v_i^\top(t) v_i(t). \tag{14}$$

Calculating the upper Dini derivative of $V(t)$ with respect to time along the solution of Eq. (12), from Condition (A_1), and note that $\sum_{i=1}^{N} v_i(t) = 0$, we can get for $t \neq t_k$,

$$\begin{aligned} D^+V(t) \quad \leq \quad & \sum_{i=1}^{N}\sum_{r=1}^{n}\Big\{\Big[-\delta_i + \frac{1}{2}\lambda_{\max}(A+A^\top) + \frac{1}{2}\sum_{s=1}^{n}(2k_{rs}^{2\varepsilon} + k_{sr}^{2(1-\varepsilon)})\Big] v_{ir}^2(t) \\ & + \frac{1}{2}\sum_{s=1}^{n} k_{sr}^{2(1-\varepsilon)} v_{ir}^2(t-\tau)\Big\} + \sum_{i=1}^{N} v_i^\top(t) \\ & \times\Big[\sum_{j=1}^{N} b_{ij}\Gamma v_j(t) + \mathrm{diag}(\delta_1,\ldots,\delta_n) v_i(t)\Big] \\ \leq \quad & -pV(t) + qV(t-\tau) + \sum_{j=1}^{n} \bar{v}_j^\top(t)(\gamma_j B + \delta_j I_N)\bar{v}_j(t), \end{aligned} \tag{15}$$

where

$$\bar{v}_j(t) = (\bar{v}_{1j}(t),\cdots,\bar{v}_{Nj}(t))^\top \in L \overset{\text{def}}{=} \big\{z = (z_1,\cdots,z_N)^\top \in R^N \mid \sum_{i=1}^{N} z_i = 0\big\},$$

from which it can be concluded that if $\gamma_j\lambda(\gamma_j) + \delta_j \leq 0$, then

$$\sum_{j=1}^{n} \bar{v}_j^\top(t)(\gamma_j B + \delta_j I_N)\bar{v}_j(t) \leq 0. \tag{16}$$

This leads to

$$D^+V(t) \leq -pV(t) + q(\sup_{t-\tau\leq s\leq t} V(s)). \tag{17}$$

On the other hand, from the construction of $V(t)$, we have

$$V(t_k) \quad = \quad (1+d_k)^2 \sum_{j=1}^{N} v_j^\top(t_k^-) v_j(t_k^-) \leq (1+d_k)^2 V(t_k^-). \tag{18}$$

It follows from the famous Halanay delay differential inequality [12] that if $\theta < \mu$ for all $t > t_0$,

$$V(t) \leq e^{-(\mu-\theta)(t-t_0)}(\sup_{t_0-\tau\leq s\leq t_0} V(s)). \tag{19}$$

This completes the proof of Theorem 1.

Remark 1.

It can be seen from (A_2) and (A_3) that robust impulsive synchronization of the controlled delayed dynamical network (4) not only depends on the coupling matrix B, the inner connecting matrix Γ, and the time delay τ, but also is heavily determined by the impulsive control gain d_k and the impulsive control interval $t_k - t_{k-1}$. Therefore, the approaches developed here further extend the ideas and techniques presented in recent literature, and they are also simple to implement in practice.

4 Application to Coupled Neural Networks

As an application of the above theoretical criteria, robust synchronization problem of a scale-free complex network composing of chaotic delayed Hopfield neural network nodes via impulsive control is discussed in this section, where numerical example is given to verify and also visualizes the theoretical results.

Example 1.

Consider a model of the controlled delayed dynamical network:

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + g(t, x_i(t), x_i(t-1)) + \sum_{j=1}^{100} b_{ij}\Gamma x_j(t), & t \neq t_k, t \geq t_0, \\ x_i(t) = (1+d_k)\big((x_i(t_k^-) - s(t)\big), & t = t_k, k = 1, 2, \cdots, \end{cases} \tag{20}$$

in which $x_i(t) = (x_{i1}(t), x_{i2}(t))^\top$, $g(t, x_i(t), x_i(t-\tau)) = B\tilde{g}(x_i(t)) + C\tilde{g}(x_i(t-1))$ with $\tilde{g}(x_i(t)) = (\tanh(x_{i1}(t)), \tanh(x_{i2}(t)))^\top$ and

$$A = \begin{bmatrix} -1.0 & 0 \\ 0 & -1.0 \end{bmatrix}, \quad B = \begin{bmatrix} 2.0 & -0.1 \\ -5.0 & 3.0 \end{bmatrix}, \quad C = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{bmatrix}, \tag{21}$$

where the synchronization state of the controlled delayed dynamical network (16) is defined as $s(t) = \frac{1}{100}\sum_{k=1}^{100} x_k(t)$.

It should be noted that the isolate delayed dynamical network

$$\dot{x}(t) = Ax(t) + g(t, x(t), x(t-1)), \tag{22}$$

is actually a chaotic delayed Hopfield neural network [10, 11, 12]. (see Fig. 1).

Now we consider an scare-free network with 100 dynamical nodes. We here take the parameters $N = 100$, $m = m_0 = 5$ and $\kappa = 3$, then the coupling matrix $B = B_{sf}$ of the SF network can be randomly generated by the B-A scale-free model [13]. In this simulation, the second-largest eigenvalue and the smallest eigenvalue of the coupling matrix B_{sf} are $\lambda_2 = -1.2412$ and $\lambda_{100} = -34.1491$ respectively.

For simplicity, we consider the equidistant impulsive interval $\tau_k - \tau_{k-1} = 0.1$ and $d_k = -0.6000\,(k \in Z^+)$. By taking $k_{11} = 2$, $k_{12} = 0.1$, $k_{21} = 5.0$, $k_{22} = 3.0$ and $\delta_r = 12\,(r = 1, 2)$, it is easy to verify that if $\gamma_1 = \gamma_2 = 6$, then all the conditions of Theorem 1 are satisfied. Hence, the the controlled coupled delayed neural network (20) will achieve robust impulsive synchronization. The simulation results corresponding to this situation are shown in Fig. 2.

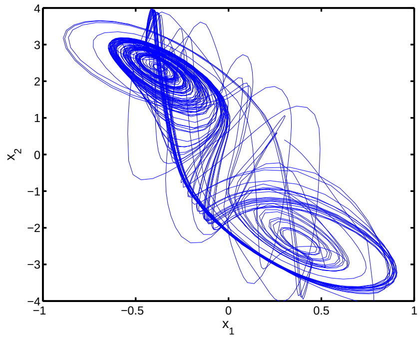

Fig. 1

Figure 1: A fully developed double-scroll-like chaotic attractors of the isolate delayed Hopfield neural network (22).

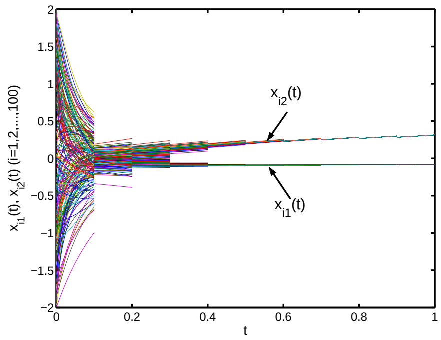

Fig. 2

Figure 2: Impulsive synchronization process of the state variables in the controlled coupled delayed neural network (20).

5 Conclusions

In this paper, we have investigated the issues of robust impulsive synchronization of coupled delayed dynamical networks. Some simple criteria for robust impulsive synchronization of such dynamical networks have been derived analytically. It is shown that the theoretical results can be applied to a typical coupled chaotic delayed Hopfield neural networks, and the numerical results are given to verify and also visualize the theoretical results.

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