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Remodelling of the Inverse DEA Model: A Prediction Tool for Not-for-profit Organizations*

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Abstract A resource allocation model (RAM) for not-for-profit organization in the framework of Data Envelopment Analysis (DEA) was introduced by Zhang and Cui ([1]). The model is an inverse formulation to the DEA model , as pointed in paper [2], and can be transformed into a parametric linear programming problem. Also a multi-objective programming problem was introduced in [2] to solve RAM when the efficiency index of the discussed organization is strictly less than one. In this paper, inadequacy of the original RAM model is discussed and then a revised new model is presented. The new model can be transformed into a multi-objective programming problem or a linear programming problem for any efficiency index value an organization has.

Keywords Resource allocation model; data Envelopment Analysis; prediction model; not-for-profit organizations; multi-objective programming

1 Introduction

Since Charnes, Cooper and Rhodes(CCR)[3] first proposed DEA method in 1978, it has become a very amateur tool for assessing the relative efficiency for not-for-profit organizations, such as government departments, military units and social service entities. The CCR ratio model can calculate the efficiency index for every DMU which reflects the existing technical structure or efficiency level. By ranking the efficiency indexes, the executives can evaluate the efficiency or inefficiency of every DMU compared with all the other same kind of DMUs. Zhang and Cui's work ([1]) was the first paper to extend DEA as a tool for resource allocation. The problem is, among a group of decision-making units or non-profit organizations, how to allocate limited resources (inputs) to a particular organization with an assumption that the organization maintains (changes) its current efficiency level with respect to other organizations according to expected increments of the service (outputs)? Or how to forecast the output change when additional investment (increment of inputs) is distributed to an organization with the same assumption? Both of the problems can be referred to as *prediction problems*.

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Denote the n DMUs by $S_1, \cdots, S_n$. Each DMU S_k has an input vector $\mathbf{x}^k \in \mathbf{R}^m$, an output vector $\mathbf{y}^k \in \mathbf{R}^s$ where $\mathbf{x}^k > \mathbf{0}, \mathbf{y}^k > \mathbf{0}$.

The problem is described as[1]: if the kth DMU plans to increase its outputs by Δy, how much additional inputs Δx should be allocated to the DMU to satisfy the efficiency expectation; or if the decision maker (DM) wants to invest increment inputs Δx to the kth DMU, how much additional outputs Δy can the kth DMU produce? Zhang and Cui transformed the original m-dimensional parameter problem to a one-dimensional parameter problem and provided an approximation algorithm to solve the problem. In the paper (Wei Quanling, et, al.[2]), the above two problems are termed in the first time as inverse DEA. The resource allocation problem based on DEA now becomes an important research topic in DEA field([4, 5, 6]).

The allocation process of RAM consists of two steps. The first step is evaluating the relative efficiency of the studied organizations. The second step then is calculating the desired changes in the input or output under the idea that the relative efficiency of the organization with new input and output remains the same.

In detail, the second step uses the DEA model in the following way: (allocation problem) knowing the relative efficiency θ_l and given an increment of the output, $\Delta\mathbf{y}^l$, find an increment of $\mathbf{x}^l$, $\Delta\mathbf{x}^l$, as small as possible such that a new DMU with $(\mathbf{x}^l + \Delta\mathbf{x}^l, \mathbf{y}^l + \Delta\mathbf{y}^l)$ still has relative efficiency ϑ_l; (investment analysis problem) If an increment of the input, $\Delta\mathbf{x}^l$, is given, we want to find an increment of $\mathbf{y}^l$, $\Delta\mathbf{y}^l$, as large as possible such that S_l with the new input and output keeps the efficiency unchanged. In this paper we mainly study the resource allocation model. For the investment analysis problem, the discussion is similar. The term *inverse DEA* is first used in paper [2] to highlight the second step of the prediction process.

In paper [2], the inverse DEA problem is transformed into and solved as a multi-objective programming problem. It is also shown that in some special cases, the inverse DEA problem can be simplified as a single-objective linear programming problem. The similar formulas can also be seen in paper [6] where the resource allocation problem is discussed. A traditional approach is proposed in both paper [2] and [6] to solve the multi-objective programming problem by allowing simultaneous proportional scaling of all the inputs or all the outputs. This paper will try to reduce the proportional assumption and exploit an algorithm to solve the inverse DEA problem.

The remainder of the paper is organized as follows. In section 2 the original resource allocation model and related computational models presented in [1] and [2] are briefly reviewed. Two linear realizations of the prediction model were introduced as computational models: parametric linear programming formulation and linear multi-objective programming formulation. But the multi-objective programming problem only works in the case $\vartheta_l < 1$. This promotes us to improve the resource allocation model. Such an improved model is presented in section 3. The improved model maintains the original economic meaning but now is well-defined. As a result, the related multi-objective programming problem now also works at $\vartheta_l = 1$. For the case that there exists a weight vector for either the input or the output, the prediction model can be realized by a linear programming problem for any value of ϑ_l.

We use $\mathbf{x} \geq_p \mathbf{y}$ to represent a set of inequalities $x_i \geq y_i, i = 1, \cdots, n$ such that there is at least one strict inequality $x_{i_0} > y_{i_0}$ existing.

2 A Prediction Model Based on DEA

The DEA model is a linear programming problem with $(\mathbf{x}^k, \mathbf{y}^k), k = 1, \cdots, n$ as coefficients of the constraint to find the relative efficiency of an assigned DMU, S_l:

$$(\mathrm{P}) \qquad \begin{array}{rl} \text{Maximize} & \mathbf{u}^T\mathbf{y}^l \\ \text{subject to} & \mathbf{w}^T\mathbf{x}^k - \mathbf{u}^T\mathbf{y}^k \geq 0,\ k = 1, \cdots, n \\ & \mathbf{w}^T\mathbf{x}^l = 1 \\ & \mathbf{w} \geq \mathbf{0},\ \mathbf{u} \geq \mathbf{0}, \end{array}$$

where $\mathbf{w} \in \mathbf{R}^m, \mathbf{u} \in \mathbf{R}^p$. Denote the optimal solution of (P) by $\mathbf{w}^l, \mathbf{u}^l$.
The dual problem of (P) is as follows:

$$(\mathrm{D}) \qquad \begin{array}{rl} \text{Minimize} & \vartheta \\ \text{subject to} & \sum_{k=1}^{n} \lambda_k \mathbf{x}^k + \mathbf{s}^- = \vartheta \mathbf{x}^l \\ & \sum_{k=1}^{n} \lambda_k \mathbf{y}^k - \mathbf{s}^+ = \mathbf{y}^l \\ & \lambda_k \geq 0,\ k = 1, \cdots, n, \\ & \mathbf{s}^+ \geq \mathbf{0},\ \mathbf{s}^- \geq \mathbf{0}, \end{array}$$

where $\lambda = (\lambda_1, \cdots, \lambda_n)^T, \mathbf{s}^+ \in \mathbf{R}^s, \mathbf{s}^- \in \mathbf{R}^m, \vartheta \in \mathbf{R}$. Denote the optimal solution by $\lambda^l = (\lambda_1^l, \cdots, \lambda_n^l)^T, \mathbf{s}^{+^l}, \mathbf{s}^{-^l}, \vartheta_l$. ϑ_l is the efficiency index of S_l.

Definition 1 *If $\vartheta_l = 1$, S_l is called weak DEA-efficient; if $\vartheta_l = 1$ and also*

$$\begin{array}{ll} \text{for any optimal solution} & (\lambda^l, \mathbf{s}^{+^l}, \mathbf{s}^{-^l}, \vartheta_l) \\ \text{there are} & \mathbf{s}^{+^l} = \mathbf{0},\ \mathbf{s}^{-^l} = \mathbf{0}, \end{array} \tag{1}$$

then S_l is called DEA-efficient. □

Based on above notation, a resource allocation model (RAM) in [1] is abstracted as follows: (The statement of investment analysis problem and the related algorithm is similar)

(RAM) *A set of DMUs has efficiency indices $\vartheta_1, \cdots, \vartheta_n$. Assign an increment, $\Delta\mathbf{y}^l \geq_p \mathbf{0}$, to the output of S_l which has efficiency index ϑ_l. Find the "smallest" additional resources, $\Delta\mathbf{x}^l$, to the input of S_l such that the resulted status of S_l remains its efficiency index unchanged.*

Let S_l be an assigned DMU whose efficiency index is $\vartheta_l \leq 1$ given by (D). As suggested in [1], we define an additional DMU S_{n+1} with input and output vectors $(\mathbf{x}^l + \Delta\mathbf{x}, \mathbf{y}^l + \Delta\mathbf{y}^l)$, where $\Delta\mathbf{x}$ is an unknown vector variable. The following extended CCR model is constructed to realize the (RAM):

(ED)
$$\begin{aligned} \text{Minimize} \quad & \vartheta \equiv \vartheta_{n+1} \\ \text{subject to} \quad & \sum_{j=1}^{n} \lambda_j \mathbf{x}^j + \lambda_{n+1}(\mathbf{x}^l + \Delta\mathbf{x}) \leq \vartheta_(\mathbf{x}^l + \Delta\mathbf{x}) \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}^j + \lambda_{n+1}(\mathbf{y}^l + \Delta\mathbf{y}^l) \geq \mathbf{y}^l + \Delta\mathbf{y}^l \\ & \lambda_j \geq 0, \ j = 1, \cdots, n+1. \end{aligned}$$

Find the 'smallest' solution $\Delta\mathbf{x}^l$ of (ED) such that the optimal value $\vartheta_{n+1} = \vartheta_l$, where ϑ_l is given by (D).

3 The Prediction Model Remodelling

Consider an extended DMU set: $\{(\mathbf{x}^1, \mathbf{y}^1), \cdots, (\mathbf{x}^n, \mathbf{y}^n), (\mathbf{x}^l + \Delta\mathbf{x}^l, \mathbf{y}^l + \Delta\mathbf{y}^l)\}$.

(RAM)$'$ *A set of DMUs has efficiency indices $\vartheta_1, \cdots, \vartheta_n$. Assign an increment, $\Delta\mathbf{y}^l \geq_p \mathbf{0}$, to the output of S_l which has efficiency index ϑ_l. Find the "smallest" additional resources, $\Delta\mathbf{x}^l$, to the input of S_l such that $S_1, \cdots, S_l, \cdots, S_n$ have their efficiency indices unchanged and the resulted status of S_l, i.e., S_{n+1} with $(\mathbf{x}^l + \Delta\mathbf{x}^l, \mathbf{y}^l + \Delta\mathbf{y}^l)$, has efficiency index ϑ_l.*

This can be put in the following detailed description. Let S_l be an assigned DMU whose efficiency index is ϑ_l. The following extended CCR model

(ED)
$$\begin{aligned} \text{Minimize} \quad & \vartheta \equiv \bar{\vartheta}_{n+1} \\ \text{subject to} \quad & \sum_{j=1}^{n} \lambda_j \mathbf{x}^j + \lambda_{n+1}(\mathbf{x}^l + \Delta\mathbf{x}) \leq \vartheta_(\mathbf{x}^l + \Delta\mathbf{x}) \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}^j + \lambda_{n+1}(\mathbf{y}^l + \Delta\mathbf{y}^l) \geq \mathbf{y}^l + \Delta\mathbf{y}^l \\ & \lambda_j \geq 0, \ j = 1, \cdots, n+1. \end{aligned}$$

has the optimal value $\bar{\vartheta}_{n+1} = \vartheta_l$. And for $k = 1, \cdots, l, \cdots, n$ the following problem

(EDk)
$$\begin{aligned} \text{Minimize} \quad & \vartheta \equiv \bar{\vartheta}_k \\ \text{subject to} \quad & \sum_{j=1}^{n} \lambda_j \mathbf{x}^j + \lambda_{n+1}(\mathbf{x}^l + \Delta\mathbf{x}^l) \leq \vartheta\mathbf{x}^k \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}^j + \lambda_{n+1}(\mathbf{y}^l + \Delta\mathbf{y}^l) \geq \mathbf{y}^k \\ & \lambda_j \geq 0, \ j = 1, \cdots, n+1. \end{aligned}$$

have optimal solution $\bar{\vartheta}_k = \vartheta_k$. Then $\Delta\mathbf{x}^l$ is the solution of the resource allocation problem.

As defined in the DEA research, the convex cone

$$T = \left\{ (\mathbf{x}, \mathbf{y}) : \begin{array}{l} \sum_{k=1}^{n} \lambda_k \mathbf{x}^k \le \mathbf{x}, \ \sum_{k=1}^{n} \lambda_k \mathbf{y}^k \ge \mathbf{y}, \\ \lambda_k \ge 0, \ k = 1, \cdots, n \end{array} \right\} \tag{2}$$

is called a *production set* associated with the problem (D). Using the concept T, The following Lemma gives a sufficient condition for a solution $\Delta\mathbf{x}^l$ of (ED) satisfying problems (EDk).

Lemma 1 *The original DMU set remains the efficiency indices with respect to the extended DMU set unchanged if* $(\mathbf{x}^l + \Delta\mathbf{x}^l, \mathbf{y}^l + \Delta\mathbf{y}^l) \in T$, *where* $\Delta\mathbf{x}^l$ *is the solution of* (ED), T *is the production set defined in (2).*

Lemma 2 *The optimal solution of* (ED) *at* $\lambda_{n+1} = \vartheta_{n+1} = 1$ *corresponds to a pair of inputs and outputs not in* T, *i.e.,* $(\mathbf{x}^l + \Delta\mathbf{x}^l, \mathbf{y}^l + \Delta\mathbf{y}^l) = (\mathbf{x}^l, \mathbf{y}^l + \Delta\mathbf{y}^l) \notin T$.

Lemma 2 says that the optimal solution with $\lambda_{n+1} = \vartheta_{n+1} = 1$ is not feasible to problem (RAM)'. With Lemma 1 the (RAM)' model can be put in a more compact form:

(RAM)* *Find the "smallest" solution* $\Delta\mathbf{x}^l$ *of problem* (ED) *such that* $\bar{\vartheta}_{n+1} = \vartheta_l$ *and* $(\mathbf{x}^l + \Delta\mathbf{x}^l, \mathbf{y}^l + \Delta\mathbf{y}^l) \in T$ *.*

The next Lemma gives an explanation: why the (RAM) model in the case $\vartheta_l < 1$ can be replaced by linear models.

Lemma 3 *If* S_l *has its efficiency index* $\vartheta_l < 1$, *then for any solution* $\Delta\mathbf{x}^l$ *of problem* (ED), $(\mathbf{x}^l + \Delta\mathbf{x}^l, \mathbf{y}^l + \Delta\mathbf{y}^l) \in T$ *is naturally satisfied.*

Corollary 1 *In the case* $\vartheta_l = 1$, *if an optimal solution of* (ED) *is reached at* $\lambda_{n+1}^* < 1$, *then* $(\mathbf{x}^l + \Delta\mathbf{x}^l, \mathbf{y}^l + \Delta\mathbf{y}^l) \in T$.

Now we return to the main problem (ED) and study the property of its feasible region:

$$\Omega = \left\{ \mathbf{f} = (\lambda, \Delta\mathbf{x}, \vartheta) : \begin{array}{l} \sum_{k=1}^{n} \lambda_k \mathbf{x}^k + \lambda_{n+1}(\mathbf{x}^l + \Delta\mathbf{x}) \le \vartheta(\mathbf{x}^l + \Delta\mathbf{x}), \\ \sum_{k=1}^{n} \lambda_k \mathbf{y}^k + \lambda_{n+1}(\mathbf{y}^l + \Delta\mathbf{y}^l) \ge \mathbf{y}^l + \Delta\mathbf{y}^l, \\ \lambda_k \ge 0, \ k = 1, \cdots, n+1, \ 0 \le \vartheta \le 1 \end{array} \right\} \tag{3}$$

Lemma 4 *It follows from Lemma 2 that if there is a feasible solution of* (RAM)*, *then it has property* $\lambda_{n+1} < \vartheta_l \le 1$. □

Lemma 5 *Let* Ω^* *be the optimal solution set of* (RAM)*, $\Omega^* = \Omega \cap \{\mathbf{f} : \ \lambda_{n+1} = 0\}$. □

In conclusion, we have the following main theorem .

Theorem 1 *Problem* (ED) *reaches its optimal solution at* $\lambda_{n+1} = 0$. *In other words,* (ED) *can be simplified as follows,*

$$(\mathrm{ED})^- \quad \begin{array}{ll} \text{Minimize} & \vartheta \equiv \vartheta_{n+1} \\ \text{subject to} & \sum_{j=1}^{n} \lambda_j \mathbf{x}^j \leq \vartheta(\mathbf{x}^l + \Delta \mathbf{x}^l) \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}^j \geq \mathbf{y}^l + \Delta \mathbf{y}^l \\ & \lambda_j \geq 0, \ j = 1, \cdots, n. \end{array}$$

It is interesting to note that the virtual DMU S_{n+1} is introduced in our first paper ([1]) to formulate a reasonable and logical prediction model. After the above discussion we know now that this virtual DMU can be neglected in the computation.

3.1 Linear Multi-Objective Programming Formulation

Consider the following linear multi-objective programming problem:

$$(\mathrm{VP}) \quad \begin{array}{ll} \mathrm{V-minimize} & (x_1^l + \Delta x_1, \cdots, x_m^l + \Delta x_m) \\ \text{subject to} & \sum_{j=1}^{n} \lambda_j \mathbf{x}^j \leq \vartheta_l(\mathbf{x}^l + \Delta \mathbf{x}) \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}^j \geq \mathbf{y}^l + \Delta \mathbf{y}^l \\ & \lambda_j \geq 0, \ j = 1, \cdots, n. \end{array}$$

where ϑ_l is the solution of problem (D). Consider the following multi-objective programming problem corresponding to (ED):

$$(\mathrm{VP})^+ \quad \begin{array}{ll} \mathrm{V-minimize} & (\Delta x_1, \cdots, \Delta x_m) \\ \text{subject to} & \sum_{j=1}^{n} \lambda_j \mathbf{x}^j + \lambda_{n+1}(\mathbf{x}^l + \Delta \mathbf{x}) \leq \vartheta_l(\mathbf{x}^l + \Delta \mathbf{x}) \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}^j + \lambda_{n+1}(\mathbf{y}^l + \Delta \mathbf{y}^l) \geq \mathbf{y}^l + \Delta \mathbf{y}^l \\ & \lambda_j \geq 0, \ j = 1, \cdots, n+1. \end{array}$$

Using the similar reasoning in Lemma 5, $(\mathrm{VP})^+$ can be simplified as

$$(\mathrm{VP})^* \quad \begin{array}{ll} \mathrm{V-minimize} & (\Delta x_1, \cdots, \Delta x_m) \\ \text{subject to} & \sum_{j=1}^{n} \lambda_j \mathbf{x}^j \leq \vartheta_l(\mathbf{x}^l + \Delta \mathbf{x}) \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}^j \geq \mathbf{y}^l + \Delta \mathbf{y}^l \\ & \lambda_j \geq 0, \ j = 1, \cdots, n. \end{array}$$

We now consider the case $\vartheta_l = 1$, i.e., the following problem

$$(\text{VP})^{**} \quad \begin{array}{ll} \text{V}-\text{minimize} & (\Delta x_1, \cdots, \Delta x_m) \\ \text{subject to} & \sum_{j=1}^{n} \lambda_j \mathbf{x}^j \leq \mathbf{x}^l + \Delta \mathbf{x} \\ & \sum_{j=1}^{n} \lambda_j \mathbf{y}^j \geq \mathbf{y}^l + \Delta \mathbf{y}^l \\ & \lambda_j \geq 0, \ j = 1, \cdots, n. \end{array}$$

Theorem 2 *Suppose that the optimal value of problem (D) is $\vartheta_l = 1$, and the outputs for this DMU are going to increase from $\mathbf{y}^l$ to $\mathbf{y}^l + \Delta\mathbf{y}^l$, $\Delta\mathbf{y}^l \geq_p \mathbf{0}$.*
(i) *Let $(\Delta\mathbf{x}^l, \bar{\lambda})$ be a weak Pareto solution of the multi-objective problem* $(\text{VP})^{**}$. *Then when the inputs of S_l are increased to $\mathbf{x}^l + \Delta\mathbf{x}^l$, the optimal value of* $(\text{ED})^{-}$ *is still ϑ_l, i.e., $\vartheta_{n+1} = 1$.*
(ii) *Conversely, let $\mathbf{x}^l + \Delta\mathbf{x}^l, \bar{\lambda}$ be a feasible solution of problem* $(\text{VP})^{**}$. *If the optimal value of problem* $(\text{ED})^{-}$ *is $\vartheta_l = 1$, then $(\Delta\mathbf{x}^l, \bar{\lambda})$ must be a weak Pareto solution of* $(\text{VP})^{**}$.

3.2 Linear Programming Formulation

In some applications it is possible to have 'price making' for the m inputs. In other words, there exists a set of weights $\mathscr{W} = (\mathscr{W}_1, \cdots, \mathscr{W}_m)^T$ associated with the m inputs. It is natural to assume that $\mathscr{W}_j > 0, j = 1, \cdots, m$.

In this case, the initial CCR model can be revised as the follows.

$$(\text{P})' \quad \begin{array}{ll} \text{Maximize} & \dfrac{\mathbf{u}^T \mathbf{y}^l}{\mathscr{W}^T \mathbf{x}^l} \\ \text{subject to} & \dfrac{\mathbf{u}^T \mathbf{y}^k}{\mathscr{W}^T \mathbf{x}^k} \leq 1, \ k = 1, \cdots, n \\ & \mathbf{u} \geq \mathbf{0}, \end{array}$$

where $\mathbf{u} \in \mathbf{R}^p$. Let $\mathscr{W}$ be normalized such that $\mathscr{W}^T \mathbf{x}^l = 1$. Using the Charnes-Cooper transformation problem $(\text{P})'$ is rewritten as

$$(\text{P})^* \quad \begin{array}{ll} \text{Maximize} & \mathbf{u}^T \mathbf{y}^l \\ \text{subject to} & \mathbf{u}^T \mathbf{y}^k \leq \mathscr{W}^T \mathbf{x}^k, \ k = 1, \cdots, n \\ & \mathbf{u} \geq \mathbf{0}, \end{array}$$

Its dual problem is

$$(\text{D})' \quad \begin{array}{ll} \text{Minimize} & \sum_{k=1}^{n} \lambda_k \mathscr{W}^T \mathbf{x}^k \\ \text{subject to} & \sum_{k=1}^{n} \lambda_k \mathbf{y}^k \geq \mathbf{y}^l \\ & \lambda_k \geq 0, \ k = 1, \cdots, n, \end{array}$$

or equivalently

$$(\mathrm{D})^* \quad \begin{aligned} \text{Minimize} \quad & \vartheta \equiv \vartheta_l \\ \text{subject to} \quad & \sum_{k=1}^{n} \lambda_k \mathscr{W}^T \mathbf{x}^k \leq \vartheta \mathscr{W}^T \mathbf{x}^l = \vartheta \\ & \sum_{k=1}^{n} \lambda_k \mathbf{y}^k \geq \mathbf{y}^l \\ & \lambda_k \geq 0,\ k = 1, \cdots, n, \end{aligned}$$

where ϑ_l is the efficiency index of S_l. The resource allocation problem is raised similarly. If the output of S_l is changed from $\mathbf{y}^l$ to $\mathbf{y}^l + \Delta\mathbf{y}^l$, what we expected for a proper increment $\Delta\mathbf{x}^l$ for the input to remain the efficiency index of S_l unchanged. A corresponding extended CCR model is

$$(\mathrm{ED})^* \quad \begin{aligned} \text{Minimize} \quad & \vartheta \equiv \vartheta_{n+1} \\ \text{subject to} \quad & \sum_{k=1}^{n} \lambda_k \mathscr{W}^T \mathbf{x}^k + \lambda_{n+1} \mathscr{W}^T (\mathbf{x}^l + \Delta\mathbf{x}^l) \leq \vartheta \mathscr{W}^T (\mathbf{x}^l + \Delta\mathbf{x}^l) \\ & \sum_{k=1}^{n} \lambda_k \mathbf{y}^k + \lambda_{n+1} (\mathbf{y}^l + \Delta\mathbf{y}^l) \geq \mathbf{y}^l + \Delta\mathbf{y}^l \\ & \lambda_k \geq 0,\ k = 1, \cdots, n+1. \end{aligned}$$

It is noted that the reasoning of Lemma 2 — Lemma 5 is still valid for problem $(\mathrm{ED})^*$. Then we only need to solve the following problem,

$$\begin{aligned} \text{Minimize} \quad & \vartheta \equiv \vartheta_{n+1} \\ \text{subject to} \quad & \sum_{k=1}^{n} \lambda_k \mathscr{W}^T \mathbf{x}^k \leq \vartheta \mathscr{W}^T (\mathbf{x}^l + \Delta\mathbf{x}^l) \\ & \sum_{k=1}^{n} \lambda_k \mathbf{y}^k \geq \mathbf{y}^l + \Delta\mathbf{y}^l \\ & \lambda_k \geq 0,\ k = 1, \cdots, n. \end{aligned}$$

There $\Delta\mathbf{x}^l$ is a vector taken as as 'small' as possible. This problem is equivalent to the following linear programming problem:

$$(\mathrm{LP}) \quad \begin{aligned} \text{Minimize} \quad & \mathscr{W}^T \Delta\mathbf{x} \\ \text{subject to} \quad & \sum_{k=1}^{n} \lambda_k \mathscr{W}^T \mathbf{x}^k \leq \vartheta_l \mathscr{W}^T (\mathbf{x}^l + \Delta\mathbf{x}^l) \\ & \sum_{k=1}^{n} \lambda_k \mathbf{y}^k \geq \mathbf{y}^l + \Delta\mathbf{y}^l \\ & \lambda_k \geq 0,\ k = 1, \cdots, n, \end{aligned}$$

where $\vartheta_l \leq 1$ is the efficiency index of S_l. The solution $\Delta\mathbf{x}^l$ of (LP) has the property described by the following theorem.

Theorem 3 *If there is a 'price making' for the inputs, then the* (RAM) *model is solved by a linear programming problem* (LP). *Each solution* $\Delta\mathbf{x}^l$ *of* (LP) *is a Pareto solution of the following linear multi-objective programming problem:*

$$(\mathrm{VP})^* \quad \begin{array}{rl} \mathrm{V-minimize} & (\Delta x_1, \cdots, \Delta x_m) \\ \text{subject to} & \sum_{k=1}^{n} \lambda_k \mathscr{W}^T \mathbf{x}^k \leq \vartheta_l \mathscr{W}^T (\mathbf{x}^l + \Delta\mathbf{x}^l) \\ & \sum_{k=1}^{n} \lambda_k \mathbf{y}^k \geq \mathbf{y}^l + \Delta\mathbf{y}^l \\ & \lambda_k \geq 0,\ k = 1, \cdots, n. \end{array}$$

4 Conclusion

A revised prediction model based on the DEA methodology is discussed in this paper. Mathematically the model is a nonlinear problem. Then linear substitutions such as linear multi-objective programming and linear programming formulation are studied.

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