The First International Symposium on Optimization and Systems Biology (OSB'07)
Beijing, China, August 8–10, 2007

A Quadratic Programming Model for Political Districting Problem

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Abstract Political Districting Problem is one of the critical issues in political elections, it can be expressed as: how to partition a state into reasonable districts for parliament election and presidential election. In this paper, the political districting problem for parliament election is modelled as a doubly weighted graph partition problem and it is formulated into quadratic programming model. Using Lingo software, we can easily find the optimal solution.

Our work can be summarized as follows: Firstly, we construct a graph for a given state H, where each node of H represents a county of the state, an edge connecting node u and v indicates that county u and v are adjacent in the map. After that, we extend H to a doubly weighted graph $G(V,E,P(v),D(e))$, in which each node represents a county of the state and the weight of a node denotes the size of population in that county; each edge $(u,v) \in E$ implies that there is a path between node u and v in graph H, and the weight of an edge (u,v) represents the length of the shortest path between u and v in graph H. Then, we convert the districting problem into a doubly weighted graph partition problem and formulate it as a quadratic programming model. Finally, we compile a software using Lingo to solve the quadratic programming problem. By using our software, we partition New York State into 29 election districts which satisfy the following conditions: Each election district has the same size of population; and the topology of each election district is continuous compactness. It is much better than the election district partition of New York in 2002.

Keywords political districting problem; doubly weighted graph partition; quadratic programming model.

1 Introduction

According to the Constitution of the United States, the number of seats each State has in the U.S. House of Representatives (usually equal with the number of election districts in each State) depends on the relative size of the State's population compared with other States. After the number of seats is assigned to the individual States, the task of drawing the new election districts is generally performed by each

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State legislature. This gives political parties a chance of drawing district lines to maximize their own advantages and win their own election. This phenomenon is called "Gerrymandering", which leads to egregious district shapes that look "unnatural" by some standards, causing large complaint and anger. So in the process of redistricting, it is highly necessary to design a fair and "simple" method, so as to make the election districts more compact and less confusing.

There are actually some existing mathematical and numerical approaches for this problem in the literature. Most of them are local search methods or heuristic algorithms such as local search methods used in Kaiser [1], Nagel [2] heuristic methods [3, 4]. To some extent, they can eliminate "Gerrymandering" phenomenon by providing well-defined steps and constraints. An implicit enumeration technique was also developed by Garfinkel and Nemhauser [5]. George et al. [6] studied the problem of determining New Zealand's electoral districts by using a location-allocation based iterative method in conjunction with geographic information system (GIS). A broad survey of political districting algorithms is given in [7].

From a mathematical point of view, the Political Districting Problem belongs to the Districting Problem (or zone design) in which n units are grouped into k zones such that some cost function is optimized subject to certain constraints on the topology of the zones, etc [8]. This problem is proved to be NP-complete [9]. Thus the enumeration technique is not feasible and it is best to be treated by some optimization methods. The constraints of the Districting Problem are very similar to that of the data clustering problem in optimization [8]. Let the set of n initial units be $X = \{x_1, x_2, \cdots, x_n\}$, where x_i is the i-th unit. Let the number of districts be k and Z_i be the set of all the units belonging to the district i. Then we have the following constraint conditions:

$$Z_i \neq \phi, i = 1, 2, \cdots, k,$$

$$Z_i \cap Z_j = \phi, i \neq j,$$

$$\bigcup_{i=1}^{k} Z_i = X.$$

There is an additional constraint in the Districting Problem, i.e., the constraint of geographical compactness which makes the problem somewhat more complicated. It requires feasible solutions to the problem to satisfy the contiguity between different units within a designed district. Geographical compactness here means that every unit in a district is connected to every other unit through units in this district and the geographical shape of every district is simple. An important optimization criterion in the Districting Problem is to avoid Gerrymandering. It is generally accepted that there are three essential characteristics which the districts should satisfy [3]: population equality, contiguity and geographical compactness. The task here is therefore to develop a method that is able to produce solutions with these characteristics. Zhou and Li [8] solve the Districting Problem by mapping the problem into a q-state Potts model.

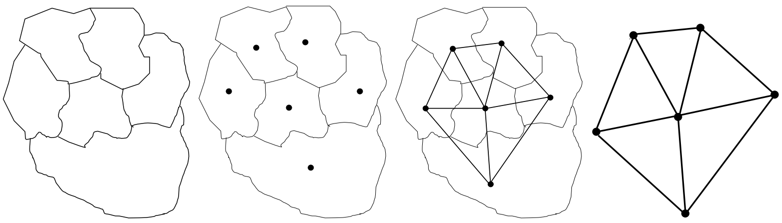

Figure 1: An illustration of the graph H extracted from the state map. The most left graph is a state map which has 6 counties; In the second graph we use a node to represent each county; In the third graph, we connect every node pair whenever they represent a pair of adjacent counties; The most right graph H is extracted from the third graph.

In contrast to the existing methods based on local search, heuristic search and implicit enumeration, in this paper, we will investigate this problem by a graph theory method. Our method for political districting problem is deterministic and can find global solutions in terms of the election district criteria. Specifically, Districting Problem is formulated as an doubly weighted graph partition problem based on which a quadratic programming model is constructed. Then we compile a software using Lingo to solve the problem. The numerical results on New York State illustrate the effectiveness of our method. This paper is organized as follows: a formulation of the redistricting problem and a description of quadratic programming model are given in Section 2. Section 3 contains the results of our numerical experiment and Section 4 makes conclusion and discussion.

2 The Quadratic Programming Model

The districting problem can be expressed as: Given a state with n counties and the size of population every county has, how to partition the state into k election districts according to some given constraints. Generally, such constraints should be reasonable and acceptable by electorates.

In our opinion, the districting problem can be formulated as a doubly weighted graph partition problem and can be converted into a quadratic programming model, in which n counties of a state are divided into q election districts such that the topology of each election district is geographical compact and every district has the same size of population.

2.1 A doubly weighted graph partition problem

Let n be the total number of counties in a state, q be the total number of districts to be partitioned in the state and p_i be the size of population in the i-th county. To solve the districting problem, we first construct an adjacent graph H for a given state, where each node denotes a county of the state. There is an edge between counties v_i and v_j if and only if counties v_i and v_j are adjacent in the map, see Fig 1 . The adjacent matrix of H is A, where

$$A_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

By using Dijkstra Algorithm, we can find the shortest path between any two nodes. Let D denote a shortest path matrix, where D_{ij} is the length of the shortest path between nodes v_i and v_j.

Then we construct a doubly weighted graph $G(V,E,P(v),D(e))$. In this graph, each node in V denotes a county of the state, and the weight $P(v_i) = p_i$ of a node v_i represents the population size in that county. For any node pair v_i and v_j, there is an edge $(v_i,v_j) \in E$. The weight D_{ij} of edge (v_i,v_j) is the length of the shortest path between v_i and v_j in graph H. If there is no path between nodes v_i and v_j in graph H, i.e., graph H is disconnected, the weight of edge (v_i,v_j) is infinity. In other words, if graph H is connected, the graph G is a doubly weighted complete graph. The doubly weighted graph can be expressed by the shortest path matrix D.

With above facts, we can formulate the Districting Problem into such a doubly weighted graph partition problem: Given a doubly weighted graph with n nodes and an integer q, find a partition of the nodes into q parts, such that the weight sum of all the nodes in each part is equal, while the total weight sum of all edges in each part is as small as possible. The equal weight sum of all the nodes in each part corresponds to the equal number of population in each district, while the minimum edge weight sum corresponds to the compactness of each election district.

2.2 The quadratic programming model

Let $x_{ij} \in [0,1]$ denote the percentage of the i-th county population grouped into the j-th election district. p_i is the size of population in the i-th county, D_{ij} is the length of the shortest path between node v_i and node v_j. q is the number of election districts the State should have. $\bar{p} = \frac{\sum_{i=1}^{n} p_i}{q}$ denotes the average size of population each election district should have. Then, the Districting Problem can be formulated by a quadratic programming model as follows:

$$\min f = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{q} p_i p_j x_{ik} x_{jk} D_{ij}$$

$$s.t. \begin{cases} \sum_{i=1}^{n} p_i x_{ik} = \bar{p} \quad k = 1,2,\cdots,q & (1) \\ \sum_{k=1}^{q} x_{ik} = 1 \quad i = 1,2,\cdots,n & (2) \\ 0 \le x_{ik} \le 1 \quad i = 1,2,\cdots,n, k = 1,2,\cdots,q & (3) \end{cases}$$

where the objective function guarantees that the shapes of the election districts are continuous and geographical compactness. Constraint (1) guarantees that every election district has an equal population size. Constraint (2) indicates that the total percentage that each county is contained in all election districts is equal to 1, which means all the people in the county take part in the parliament election of the state. Constraint (3) indicates the bounds of the variables. If the i-th county is completely grouped to the k-th election district, $x_{ik} = 1$, while if the i-th county is partly contained in the k-th election district, x_{ik} is equal to a value less than 1. Similarly, if the i-th county does not belong to the k-th election district at all, $x_{ik} = 0$.

2.3 Remark

In the quadratic programming model, the constraints guarantee that every election district has the same size of population, while the objective function achieves the compactness of the election districts. In our paper, we use the adjacent matrix H rather than the real distance of two nodes, because the density of population in different counties is different. So we think the adjacent relationship can express the compactness much better.

The restriction of the population equality will sometimes destroy the contiguity and compactness in districting, so we can allow the size of population in each election district to be approximately equal.

Based on the adjacent matrix H, we can find the shortest distance D_{ij} between any two counties by Dijkstra Algorithm and use Matlab software to implement the algorithm.

3 Simulation Results

We have constructed the quadratic programming model of Districting problem. In this section, we will test the effectiveness of our method by the New York State districting problem.

3.1 Constructing and solving the quadratic programming model

According to congressional districting of New York State in 2002, the number of the districts in New York is $q = 29$, where each district will elect its own seats. New York State totally has $n = 62$ counties (a county corresponds to a node in the doubly weighted graph). The total population size is 18976457 and the average population size of each district is $\bar{p} = 654360.6$.

Let $0 \le x_{ik} \le 1, i = 1,2,\cdots 62, k = 1,2,\cdots,29$ denote the population percentage of the i-th county grouped into the k-th congressional district. p_i is the size of population in the i-th county and D_{ij} is the length of the shortest path between node v_i and node v_j. In this model, $n = 62$, $q = 29$. Therefore, the quadratic programming model for districting New York State is as follows:

$$\min f = \sum_{i=1}^{62}\sum_{j=1}^{62}\sum_{k=1}^{29} p_i p_j x_{ik} x_{jk} D_{ij}$$

$$s.t. \begin{cases} \sum_{i=1}^{62} p_i x_{ik} = \bar{p} \quad k = 1,2,\cdots,q \\ \sum_{k=1}^{29} x_{ik} = 1 \quad i = 1,2,\cdots,n \\ 0 \le x_{ik} \le 1 \quad i = 1,2,\cdots,n, k = 1,2,\cdots,q \end{cases}$$

where $\bar{p} = 654360.6$.

Actually, according to the Constitution of the United States, population constraint is not so strict, and the deviation from average population is allowed if it is no more than δ ($\delta \leq 0.03$). Base on this fact, we can relax the first constraint and give the following model.

$$\min f = \sum_{i=1}^{62}\sum_{j=1}^{62}\sum_{k=1}^{29} p_i p_j x_{ik} x_{jk} D_{ij}$$

$$s.t. \begin{cases} \sum_{i=1}^{62} p_i x_{ik} \leq (1+\delta)\bar{p} \quad k = 1,2,\cdots,q \\ \sum_{i=1}^{62} p_i x_{ik} \geq (1-\delta)\bar{p} \quad k = 1,2,\cdots,q \\ \sum_{k=1}^{29} x_{ik} = 1 \quad i = 1,2,\cdots,n \\ 0 \leq x_{ik} \leq 1 \quad i = 1,2,\cdots,n, k = 1,2,\cdots,q \end{cases}$$

Clearly, this model is a quadratic programming problem with 1798 variables and 182 constraints. We compile a program using LINGO to solve this problem.

According to the running results of the Lingo software, we can find the corresponding counties each congressional district contains. We can list all the congressional districts of New York State and all the counties in each congressional district, deviation from average population and so on.

3.2 Drawing the congressional districts map and comparing with the former districting map

According to the result, we can approximately draw out the boundaries of all congressional districts on New York State map, and compare it with the real congressional districts of 2002. In the process of drawing boundaries, we assume that the density of a county is a constant. So we can divide a county according to the area.

Firstly, for a given congressional district, we find the counties that it contains and draw the boundary line including them. For a county that partially belongs to a congressional district, i.e. the value of the corresponding variable is less than 1, we will only cut a certain percent area of the county. For example, if $x_{4,3} = 0.3$, which means that 30 percent of the 4-th county's population belongs to the 3rd congressional district, we only need to cut 30 percent area of the 4-th county and merge it into the 3rd congressional district. For the purpose of maintaining district compactness, we always cut this county from the side adjacent to the congressional district it is partially contained in.

The congressional district map of New York is illustrated in the Figure 2, followed by the former New York congressional district map in Figure 3. By comparing with these two maps, we can find that our result is much better and acceptable than the former one, since each congressional district is more compact than the former ones. There are no egregious district shapes that look "unnatural" in the districts partitioned by our methods. Such a districting result could reduce large complaints and angers of electorates.

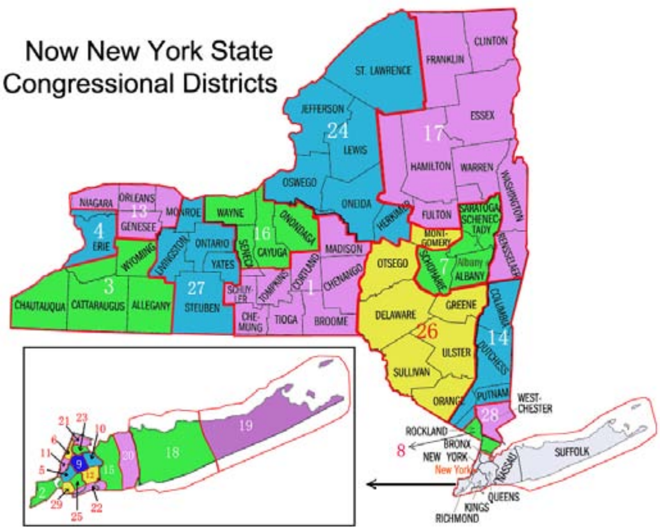

Figure 2: An illustration of New York State districting result by our model, the 29 congressional districts are drawn in different colors.

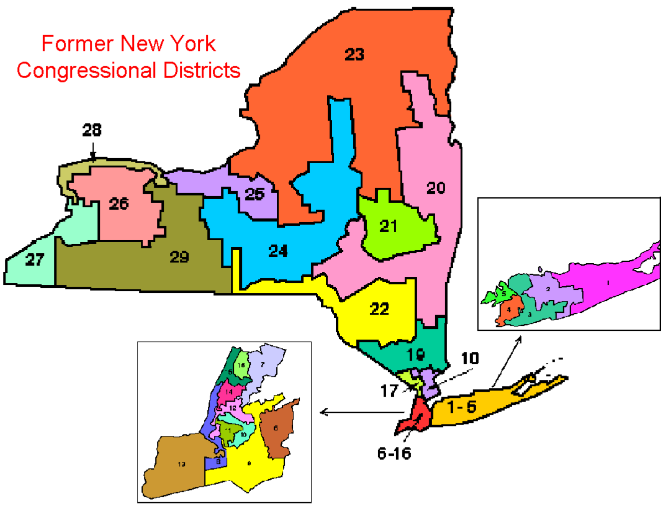

Figure 3: An illustration of New York State districting result of 2002, the 29 congressional districts are drawn in different colors.

4 Conclusion and Discussion

In this paper, we investigate the political districting problem. First, the districting problem is transformed into doubly weighted graph partition problem, then we formulate and solve it by a quadratic programming model. By using Lingo software, we can easily find the optimal solution. Our method is deterministic and globally optimal.

In the quadratic programming model,if we restrict every election district to have an equal population size, some counties may be partitioned into two or more different election districts. This will destroy the compactness of election districts. For example, if we restrict the population size of every district of New York to be equal, then the result will not be so good as that in Section 3. On the other hand, if we relax the restraints, the result may be much better in terms of compactness. Any way, since we use continuous variables $x_{ik} \in [0,1]$ instead of 0-1 variables to express the percentage of the i-th county's population grouped into the k-th election district, we can always find an optimal solution.

Since we incorporate the geographical compactness into the quadratic programming model, the simulation result of the model is convincing and will be well accepted by electorates. On the other hand, the objective function of the model highly embodies geographical compactness, so the solution of the model is more practical and the election districts can be drawn out compactly. Furthermore, the quadratic programming model can also be used in various other real districting problems, such as school districting, home-care districting [12], emergency service systems districting [13], electrical power districting [10], police district design [11], etc.

Acknowledgements This work is partly supported by Funding Project for Academic Human Resources Development in Institutions of Higher Learning Under the Jurisdiction of Beijing Municipality (PHR(IHLB)), and K.G.Wang Education Foundation Hong Kong.

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