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Harvesting from a Population in a Stochastic Crowded Environment with Harvesting Cost

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Abstract We study the problem on optimal harvesting from a population in a stochastic crowded environment with harvesting cost. In this problem, the size of the population satisfy the stochastic logistic differential equation, which serves as a model for population growth in a stochastic environment with finite carrying capacity when harvesting is exercised. We consider the following problem: What harvesting strategy maximizes the expected discounted total harvested profit? We formulate this as a stochastic control problem and obtain the optimal harvesting strategy and the corresponding optimal harvesting profit function explicitly.

Keywords optimal harvesting problem; stochastic control; stochastic logistic differential equation; harvesting cost; local time

1 Introduction

Theory of stochastic control is widely applied in many fields such as finance, insurance, bioeconomics, etc. Many kinds of models on control problems have been put forward to meet the needs of practice. In this paper, we study the optimal harvesting strategy when there is harvesting cost using stochastic analysis and the classical theory of stochastic control.

Bioeconomic resource models incorporating random fluctuations in either population size or model parameters have been the subject of much interest. The problem of optimally harvesting is extremely important in mathematical bioeconomics and has been widely studied. The canonical example is asking how to get the most out of a logistic growth model. A classic model for population growth in a stochastic crowded environment proposed by E. M. Lungu and B. Øksendal [1] is to represent the size of the population at time t as the solution of the *stochastic logistic differential equation*

$$dX_t = rX_t(K-X_t)dt + \alpha X_t(K-X_t)dW_t, t \geq 0; X_0 = x > 0, \tag{1}$$

where $r \in \boldsymbol{R}$, $K > 0$, $\alpha \in \boldsymbol{R}$, and $x \geq 0$ are given constants. The constant r is a measure of the quality of the environment (for this population). The constant K is called the carrying capacity of the environment. The constant α is a measure of the size of the noise in the system. W_t denotes one-dimensional standard Brownian motion and dW_t is the Itô differential. See, for example, [2] for more information about stochastic differential equations.

An alternative model, studied by Alvarez and Shepp [3] and later by Myhre [4] generalizes the logistic model by adding a noise term where relative uncertainty is constant:

$$dX_t = rX_t(1 - X_t/K)dt + \alpha X_t dW_t, t \geq 0; X_0 = x > 0, \tag{2}$$

Alvarez [5] has also analyzed a class of models described by

$$dX_t = r(X_t)X_t dt + \alpha(X_t)dW_t, t \geq 0; X_0 = x > 0, \tag{3}$$

N. C. Framstad [6] studied a model described by

$$dX_t = r(X_t)X_t dt + \alpha(X_t)X_t dW_t + \int zM(dt,dz), t \geq 0; X_0 = x > 0, \tag{4}$$

where M is the centered random measure.

All the aforementioned works are concerned with the problem of maximizing the expected discounted total harvest.

In this paper, we emphasize that we will instead be introducing an important factor—harvesting cost to the model. We study the following question: Suppose we have a population (e.g., a fish population in a lake) whose size X_t at time t is described by the stochastic differential Equation (1). At what times–and how much–should we harvest from the population to maximize the expected discounted total harvested profit? We formulate this as a stochastic control problem and obtain optimal harvesting strategy and optimal profit function. We find the solution by using variational inequalities.

2 Optimal harvesting model with harvesting cost

Assume to be given a filtered probability space ($\boldsymbol{\Omega}$, $\mathscr{F}$, $(\mathscr{F}_t)_{t\geq 0}$, $\boldsymbol{P}$) satisfying the usual conditions. Suppose that, when we do not make any interventions, the population that we consider has size X_t at time t given by the stochastic logistic equation

$$dX_t = rX_t(K - X_t)dt + \alpha X_t(K - X_t)dW_t, t \geq 0; X_0 = x > 0. \tag{5}$$

The coefficients of this equation, $b(x) = rx(K - x)$ and $\sigma(x) = \alpha x(K - x)$ do not satisfy the linear growth conditions $|b(x)| \leq C(1+|x|)$ for all x and $|\sigma(x)| \leq C(1+|x|)$ for all x for some constant C, and therefore it is not straightforward from the general theory (See, for example, Chapter 5 in [2]) that Equation (1) has a global solution; that is, a solution defined for all $t \geq 0$. Nevertheless, it can be proved (See Section 2 in [1]) that if either$r \geq 0$, (and $x \geq 0$ is arbitrary) or $0 \leq x \leq K$, (and $r \in \boldsymbol{R}$ is arbitrary), then the stochastic logistic equation (1) has a unique, global strong solution X_t defined for all $t \geq 0$.

In the deterministic case, the solution is well known. See, for example, Section 2.5 in [7] and the references therein.

The stochastic population [model; Eq. (5)] is one of several possible stochastic versions of the classical Verhulst model for deterministic population growth in a crowded environment. It can be arrived at this model by introducing white noise in the corresponding deterministic logistic Equation and using the corresponding Itô interpretation of the integral.

Suppose that the population is, say, a fish population in a lake and that we decide to harvest (fish) from the population.

We denote by h_t the cumulative harvested amount up to and including time t, then the size of the harvested population X_t^h will satisfy the equation

$$dX_t^h = rX_t^h(K - X_t^h)dt + \alpha X_t^h(K - X_t^h)dW_t - dh_t, t \geq 0; X_0^h = x > 0. \tag{6}$$

If the discounting exponent is ρ, then the expected discounted total harvested profit from time 0 on is given by

$$J^h(x) = E_x[\int_{0-}^{T} e^{-\rho t}(\lambda dh_t - c(X_t^h)dt)], \tag{7}$$

where E_x denotes the expectation with respect to the probability law P_x of $(X_t^h)_{t\geq 0}$ starting at x at time t = 0, while $T = \inf\{t > 0: X_t^h = 0\}$ is the time when the population dies out (if it does), λ denotes the value of per unit harvested population, $c(\cdot)$ is a nonnegative function representing the harvesting cost, which has two continuous derivatives and depends on the size of the harvested population X_t^h, and satisfies

$$E_x[\int_0^{\infty} e^{-\rho t} c(X_t^h)dt)] < \infty \tag{8}$$

and

$$c''(x) \geq -2r\lambda \quad for \quad x > 0. \tag{9}$$

We call a process $\{h_t , t \geq 0\}$ an admissible control strategy if it is a right continuous, absolutely continuous, nonnegative, nondecreasing, and $\mathscr{F}_t$-adapted process and satisfies (8). We denote by $\boldsymbol{H}$ all the admissible control strategies.

This leads to the following problem:

Find the function $\Phi(x)$ and an optimal harvesting strategy $h^* \in \boldsymbol{H}$ such that

$$\Phi(x) = \sup_{h \in H} J^h(x) = J^{h*}(x). \tag{10}$$

3 A verification theorem

We denote by $\hat{X}_t^h$ the reflection downward of X_t^h when it reaches a certain (fixed) level x* and by ξ_t the local time of $\{X_t^h\}$ at x*. Assume $X_0^h = x \leq x^*$ then $\hat{X}_t^h$ and ξ_t can be obtained as the solution of the Skorohod stochastic differential equation [8]

$$d\hat{X}_t^h = r\hat{X}_t^h(K - \hat{X}_t^h)dt + \alpha \hat{X}_t^h(K - \hat{X}_t^h)dW_t - d\xi_t \tag{11}$$

Then $\hat{X}_t^h$ and ξ_t have the following properties: $\hat{X}_t^h \le x^*$, for all $t \ge 0$; ξ_t is nondecreasing, increasing only when $\hat{X}_t^h = x^*$; $\hat{X}_0^h = x \le x^*$ and $\xi_0 = 0$.

Theorem 1 Suppose that we can find $\psi \in \boldsymbol{C}^2(\boldsymbol{R})$, $\psi \ge 0$, such that

$$\psi'(x) \ge \lambda \quad for \quad all \quad x > 0 \tag{12}$$

$$L\psi(x) := r\psi'(x)x(K-x) + \frac{1}{2}\alpha^2\psi''(x)x^2(K-x)^2 - \rho\psi(x) \le c(x) \quad for \quad all \quad x > 0 \tag{13}$$

then

$$\psi(x) \ge \Phi(x) \quad for \quad all \quad x > 0 \tag{14}$$

In addition, assume that there exist a constant C and a positive constant x^* such that

$$L\psi(x) = c(x) \quad for \quad x < x^* \tag{15}$$

$$\psi(x) = \lambda(x - x^*) + C \quad for \quad x \ge x^* \tag{16}$$

Define the following control

$$\hat{h}_t = \begin{cases} (x - x*)^+ & for \quad t = 0 \\ \xi_t & for \quad t > 0 \end{cases} \tag{17}$$

where ξ_t is the local time of $\hat{X}_t$ given by Equation (11). Suppose that

$$E_x[\psi(X^{\hat{h}}_{\hat{T}_R})] \to 0 \quad as \quad R \to \infty, \tag{18}$$

where $T_R = T \wedge R \quad \wedge \inf\{t \ge 0\text{: } X_t^{\hat{h}} \ge R\}$, then

$$\psi(x) = \Phi(x) = J^{h*}(x) \quad for \quad all \quad x \quad > 0 \tag{19}$$

and $h^* := \hat{h}$ is optimal.

The proof of Theorem 1 is given in Appendix.

4 Optimal harvesting strategy

In this section, we will give the optimal harvesting strategy and the corresponding optimal harvesting profit explicitly.

It is clear that to accomplish this task, it suffices to determinate the function $\psi(x)$ that satisfies (12)-(18).

It requires us to find a function $f(x)$ and two constants x^*, C that satisfy

$$Lf(x) = c(x) \quad for \quad x < x^*$$

$$f'(x) \ge \lambda \quad for \quad x < x^*; f'(x^*) = \lambda$$

$$f''(x^*) = 0$$

$$r\lambda x(K-x) - \rho[\lambda(x - x^*) + C] \le c(x) \quad for \quad x \ge x^* \tag{20}$$

$$E_x[f(X^{\hat{h}}_{\hat{T}_R})] \to 0 \quad as \quad R \to \infty \tag{21}$$

After it is done, we let

$$\psi(x) = \begin{cases} f(x) \; for \; x < x* \\ \lambda(x - x*) + C \; for \; x \geq x* \end{cases}, \tag{22}$$

then from Theorem 1 we can conclude that $\psi = \Phi$ and we have an explicit description of the optimal harvesting strategy h*, which is given in (17).

Assume from now on that $\rho < rK$ and let $\theta = \frac{1}{2}[1 - \frac{2r}{\alpha^2 K} + \sqrt{(\frac{2r}{\alpha^2 K} - 1)^2 + \frac{8\rho}{\alpha^2 K^2}}]$, then $0 < \theta < 1$.

Then from Section 4 in [1] and using the method describeed in Chapter 2 in [9] we can find a function $f_0(x)$ as a particular solution of the equation

$$Lf(x) = c(x) \tag{23}$$

and we know

$$f(x) := f_0(x) + Ax^{\theta} g(x) \tag{24}$$

is strictly increasing in (0, K) and always a solution of Equation (23), where $A > 0$ is an arbitrary constant and $g(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_0 = 1$, and the function $h(x) := f'(x)$ must have a minimum in (0, K). Therefore there exists at least one point $x \in (0, K)$ such that

$h'(x) = f''(x) = 0$.

Define $x^* = \inf\{x > 0 : f''(x) = 0\}$, then $f''(x^*) = 0$, and

$$f''(x*) < 0, \quad for \quad all \quad x \in (0, x*) \tag{25}$$

and by virtue of we can choose the constant $A > 0$ in Equation (24) such that $f'(x^*) = \lambda$.

Then by inequality (25), we have

$f'(x) > \lambda$ for $x \in (0, x^*)$.

Finally let $C = f(x^*)$, from (25), we get

$c(x) = \boldsymbol{L}f(x) = rf'(x)x(K - x) + \frac{1}{2}\alpha^2 f''(x)x^2(K - x)^2 - \rho f(x)$

$< rf'(x)x(K - x) - \rho f(x)$ for $x < x^*$

Hence, if we put

$F(x) = r\psi'(x)x(K - x) - \rho\psi(x) - c(x)$ for $x > 0$,

we have

$F(x) > 0$ for $x < x^*$ and $F(x^*) = 0$.

Therefore

$$F'(x*) \leq 0. \tag{26}$$

that is,

$rf''(x^*)x^*(K-2x^*)+rf'(x^*)(K-2x^*)-\rho f'(x^*)-c'(x^*)\leq 0.$

Because $f''(x^*)=0$ and $f'(x^*)=\lambda$, this gives

$$x^* \geq \frac{K}{2}-\frac{\rho}{2r}-\frac{c'(x*)}{2r\lambda} \tag{27}$$

Note that

$F'(x) = r\lambda(K-2x)-\rho\lambda-c'(x)$ for $x \geq x^*$,

$F''(x) = -2r\lambda-c''(x)$ for $x \geq x^*$,

then combining (9) and (26) we get $F'(x) \leq F'(x^*) \leq 0$ for $x \geq x^*$.

Then we conclude that

$F(x) \leq F(x^*) = 0$ for $x \geq x^*$.

and Equation (20) is verified.

Finally, let $\Omega_0 = \{\omega: T(\omega)=\infty\}$, similar to [1], we can verify condition (21) for both $\omega \in \Omega_0$ and $\omega \notin \Omega_0$.

Thus the function $\psi(x)$ defined by (22) satisfies all the requirements of Theorem 1 and we conclude that ψ coincides with the value function Φ.

Obviously the harvesting strategy $\hat{h}$ defined by (17) is an optimal harvesting strategy.

5 Conclusions

The problem on optimal harvesting from a population in a stochastic crowded environment with harvesting cost considered in this paper can be regarded as a mathematical formulation of the problem of *finding the best profitable and sustainable harvesting strategy under uncertainty*, which is to maximize the expected discounted total harvested profit. We formulate this as a stochastic control problem and obtain the optimal harvesting strategy and the corresponding optimal harvesting profit function explicitly. We have proved that the optimal strategy is as follows. Wait until the population reaches the size x^* before starting the harvesting (or harvest the population down to x^* immediately if it initially is above x^*); and, from then on, harvest only when $X_t \geq x^*$, and then exactly the (minimum) amount needed to prevent the process from exceeding the level x^*. See Figure 1.

It can be seen through simple computation that the population level at which harvesting is done should be higher when there is uncertainty than when there is no uncertainty.

Appendix

Proof of Theorem 1. Choose an arbitrary harvesting strategy $h \in \boldsymbol{H}$ and assume that $\psi \in \boldsymbol{C}^2(\boldsymbol{R})$, $\psi \geq 0$, and satisfies Equations (12) and (13). Then by Ito's formula for semimartingales–see, for example, Chapter II in [10]–we have, by Equation (6),

$$\begin{aligned} e^{-\rho TR}\psi(X^h_{TR}) &= \psi(x)+\int_0^{T_R} e^{-\rho t}L\psi(X^h_t)dt+\int_0^{T_R}\alpha e^{-\rho t}\psi'(X^h_t)X^h_t(K-X^h_t)dW_t \\ &- \int_0^{T_R} e^{-\rho t}\psi'(X^h_t)dh_t+\sum_{0<t\leq T_R} e^{-\rho t}[\Delta\psi(X^h_t)+\psi'(X^h_{t-})\Delta h_t], \end{aligned} \tag{A.1}$$

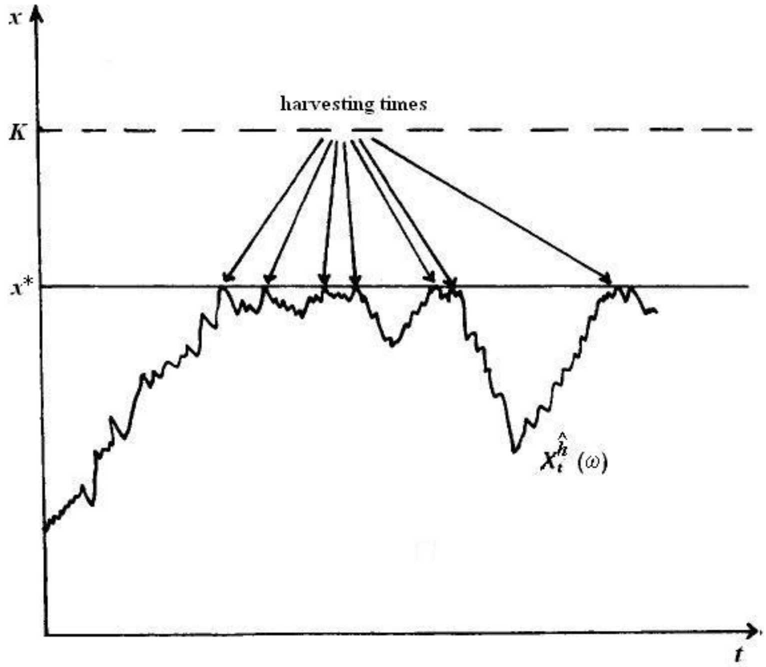

Figure 1: The optimal harvesting rule and the corresponding population process (reflection at $x = x^*$).

where $\Delta\psi(X_t^h) = \psi(X_t^h) - \psi(X_{t-}^h)$.

With the use of Equation (13), this gives

$$\begin{aligned} E_x[e^{-\rho TR}\psi(X_{TR}^h)] &= \psi(x) + E_x[\int_0^{T_R} e^{-\rho t} L\psi(X_t^h)dt] - E_x[\int_0^{T_R} e^{-\rho t}\psi'(X_t^h)dh_t] \\ &\quad + E_x[\sum_{0<t\leq T_R} e^{-\rho t}[\Delta\psi(X_t^h) + \psi'(X_{t-}^h)\Delta h_t] \\ &\leq \psi(x) + E_x[\int_0^{T_R} e^{-\rho t} c(X_t^h)dt] - E_x[\int_0^{T_R} e^{-\rho t}\psi'(X_t^h)dh_t] \\ &\quad + E_x[\sum_{0\leq t\leq T_R} e^{-\rho t}[\Delta\psi(X_t^h) + \psi'(X_{t-}^h)\Delta h_t] - E_x[\psi'(x)\Delta h_0] \end{aligned} \quad (A.2)$$

Let h_t^c denote the continuous part of h_t; that is, $h_t^c = h_t - \sum_{0<u\leq t} \Delta h_u$.

Then Equation (A.2) can be written as

$$\begin{aligned} E_x[e^{-\rho TR}\psi(X_{TR}^h)] &\leq \psi(x) + E_x[\int_0^{T_R} e^{-\rho t} c(X_t^h)dt] - E_x[\int_0^{T_R} e^{-\rho t}\psi'(X_t^h)dh_t^c] \\ &\quad + E_x[\sum_{0\leq t\leq T_R} e^{-\rho t}\Delta\psi(X_t^h)] \end{aligned} \quad (A.3)$$

On the other hand, by the mean value property, we have

$$\Delta\psi(X_t^h) = \psi'(\eta_t)\Delta X_t^h = -\psi'(\eta_t)\Delta h_t \quad (A.4)$$

for some $\eta_t \in (X_{t-}^h, X_t^h)$.

Hence, combining Equations (A.3) and (A.4) with condition (12), we get

$E_x[e^{-\rho TR}\psi(X_{TR}^h)] \leq \psi(x) - E_x[\int_{0-}^{T_R} e^{-\rho t}(\lambda \mathrm{d}h_t - c(X_t^h)\mathrm{d}t)]$

Hence

$$\begin{aligned} \psi(x) &\geq \overline{\lim_{R\to\infty}} E_x[e^{-\rho TR}\psi(X_{TR}^h)] + E_x[\int_{0-}^{T} e^{-\rho t}(\lambda dh_t - c(X_t^h)dt)] \\ &\geq E_x[\int_{0-}^{T} e^{-\rho t}(\lambda dh_t - c(X_t^h)dt)] \end{aligned} \tag{A.5}$$

Because this holds for arbitrary $h \in \boldsymbol{H}$, we conclude that $\psi(x) \geq \Phi(x)$, which is Equation (14).

Now Let's prove the rest of Theorem 1.

First we suppose that $x < x^*$, then we have $\hat{h}_t = \xi_t$, $t \geq 0$, applying calculations (A.1)-(A.5) to $h = \hat{h}$, we get equality everywhere. Therefore,

$E_x[e^{-\rho \hat{T}_R}\psi(X_{\hat{T}_R}^{\hat{h}})] = \psi(x) + E_x[\int_0^{\hat{T}_R} e^{-\rho t}c(X_t^{\hat{h}})\mathrm{d}t] - E_x[\int_{0-}^{\hat{T}_R}\lambda e^{-\rho t}\mathrm{d}\hat{h}_t]$

Then we have

$\psi(x) = \lim_{R\to\infty} E_x[e^{-\rho \hat{T}_R}\psi(X_{\hat{T}_R}^{\hat{h}})] + E_x[\int_{0-}^{T} e^{-\rho t}(\lambda \mathrm{d}\hat{h}_t - c(X_t^{\hat{h}})\mathrm{d}t)]$

$= E_x[\int_{0-}^{T} e^{-\rho t}(\lambda \mathrm{d}\hat{h}_t - c(X_t^{\hat{h}})\mathrm{d}t)] = J^{\hat{h}(x)}$

by condition (17).

Next we suppose that $x \geq x^*$, then we have

$J^{\hat{h}(x)} = \lambda(x - x^*) + E_{x^*}[\int_0^T e^{-\rho t}(\lambda \mathrm{d}\hat{h}_t - c(X_t^{\hat{h}})\mathrm{d}t)]$

$= \lambda(x - x^*) + \psi(x^*) = \psi(x)$

Then we know $\psi(x) = J^{\hat{h}}(x)$ for all $x > 0$

Combining this with Equation (14), we get Equation (19) and hence $h^* := \hat{h}$ is optimal.

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