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Study on Winning Probabilities of Two-versus-Two Stochastic Duel with Searching

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Abstract The stochastic duel theory comes from applying stochastic process to Lanchester equation. The stochastic duel with searching is a kind of duel, not less than one of whose two contestants are stealthily. When on the assumption that the time intervals of two contestants is the same negative exponential distribution(NED), the stochastic duel is a Markov process which is continuous in time and discrete in states. The winning probability formulas of two-versus-two stochastic duel with searching are deduced using state transfer figure and the property of Laplace transformation. Finally the simulations and calculations are done in a true combat environment. The results of the winning probabilities formulas are exact probabilistic solutions, which can be used to quantificationally evaluate operation efficiency of more realistic small-to-moderate-size firefight models.

Keywords stochastic duel with searching; Markov process; winning probability; state transfer figure

1 Introduction

Trevor Williams and C. J. Ancher, Jr. published their paper [1] on "Operations Research" in 1965, which applying stochastic process to Lanchester equation. Since then stochastic duel gradually becomes one of the important research subject of military operations research. The stochastic duel with searching is a kind of duel, not less than one of whose two contestants are stealthily, the other needs to search and then attack the contestant. The stochastic duel with searching was preliminary studied in papers [2,3] and the winning probabilities formulas of one-versus-one was given. We will deduce the winning probabilities formulas of two-versus-two, one simple example of many-versus-many stochastic duel with searching, which can be used to quantifacationally evaluate operation efficiency of more realistic small-to-moderate-size firefight models.

2 Basic conception & assumption

In the stochastic duel with searching, two contestants, A and B, fire at each other at uncertain intervals. B(contain many units) is stealthily and A (contain many units) needs to firstly search B and then attack. We take two-versus-two stochastic duel with searching for example and deduce the winning probabilities formulas of two contestants.

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Now we assume that the time A searches B is a positive random variable ξ and is negative exponential distribution(NED), whose parameter is θ. Also the shot time intervals of two contestants η_A and η_B are the same negative exponential distributions(NED), whose parameters are r_A and r_B. That is

$$f_\xi(t) = \begin{cases} \theta e^{-\theta t} & t \geq 0 \\ 0 & t < 0 \end{cases}, f_{\eta_A}(t) = \begin{cases} r_A e^{-r_A t} & t \geq 0 \\ 0 & t < 0 \end{cases}, f_{\eta_B}(t) = \begin{cases} r_B e^{-r_B t} & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{1}$$

p_A is the constant single-shot kill probability of A attacks B and p_B is the constant single-shot kill probability of B attacks A from round to round. Now in one-versus-one stochastic duel we define $\rho_A = p_A r_A$ as kill rate of A versus B. Also we define $\rho_B = p_B r_B$ as kill rate of B versus A. Finally we define θ as search rate of A. In paper [3] we conclude that the kill rate of A is $m\rho_A$ in many-versus-one stochastic duel. When the two-versus-two stochastic duel becomes two-versus-zero or zero-versus-two, it implies that A wins or B wins. $P_A(2:2)$ and $P_B(2:2)$ are winning probabilities of A and B in two-versus-two stochastic duel.

3 Winning probability formulas of two-versus-two stochastic duel with searching & its deduction

Winning probability formulas of two contestants, A and B in two-versus-two stochastic duel with searching are:

$$\begin{aligned} P_A(2:2) = & \frac{2\theta\rho_A^2}{(\theta+2\rho_B)(\rho_A+\rho_B)(2\rho_A+\rho_B)} + \frac{2\theta\rho_A(\rho_A+\theta)\rho_B}{(\theta+2\rho_B)(\theta+\rho_B)(2\theta+\rho_A+2\rho_B)(\rho_A+\rho_B)} \\ & + \frac{2\theta\rho_A^2\rho_B}{(\theta+2\rho_B)(\rho_A+2\rho_B)(2\theta+\rho_A+2\rho_B)(\rho_A+\rho_B)} \\ & + \frac{3\theta\rho_A^2\rho_B}{(\theta+2\rho_B)(\rho_A+\rho_B)(\rho_A+2\rho_B)(2\rho_A+\rho_B)} \end{aligned} \tag{2}$$

$$\begin{aligned} P_B(2:2) = & \frac{4\theta\rho_B^2}{(\theta+2\rho_B)(2\theta+\rho_A+2\rho_B)(\rho_A+2\rho_B)} + \frac{2\theta\rho_B^2}{(\theta+2\rho_B)(\rho_A+\rho_B)(\rho_A+2\rho_B)} \\ & + \frac{4\rho_B^2+2(\rho_A^+\theta)\rho_B^2}{(\theta+2\rho_B)(2\theta+\rho_A+2\rho_B)(\theta+\rho_B)} + \frac{2\theta(\rho_A+\theta)\rho_B^2}{(\theta+2\rho_B)(\theta+\rho_B)(2\theta+\rho_A+2\rho_B)(\rho_A+\rho_B)} \\ & + \frac{2\theta\rho_A\rho_B^2}{(\theta+2\rho_B)(\rho_A+2\rho_B)(2\theta+\rho_A+2\rho_B)(\rho_A+\rho_B)} \\ & + \frac{3\theta\rho_A\rho_B^2}{(\theta+2\rho_B)(\rho_A+\rho_B)(\rho_A+2\rho_B)(2\rho_A+\rho_B)} \end{aligned} \tag{3}$$

Proof The state transfer figure of two-versus-two stochastic duel with searching is illustrated in Fig.1(take no account of search strategy and attack strategy of two contestants). There are eleven states in the whole stochastic duel and the state $\overline{2}:2$ is the initial state which denotes that A hasn't found B while B has already found A. The states $2:2$, $2:1$ and $1:1$ denote that the two contestants have found each other and become two-versus-twoąćtwo-versus-one and one-versus-one stochastic duel. The state $\overline{1}:2$ denotes that one unit of A has been destroyed and the other unit continues to search B. The state $1:2$ denotes that A has found all of B after searching and becomes one-versus-two stochastic duel. The state $\overline{1}:1$ denotes that A has found one unit of B and destroyed it while continues to search another unit of B. The states $0:2$ and $0:1$ denote that B has already destroyed all units of A and won. The states $2:0$ and $1:0$ denote that A has already destroyed all units of B and won. All the transfer rates from one state to another are marked in the Fig.1 such as θ, $2\rho_A$, $2\rho_B$, $\rho_A+\theta$ and so on.

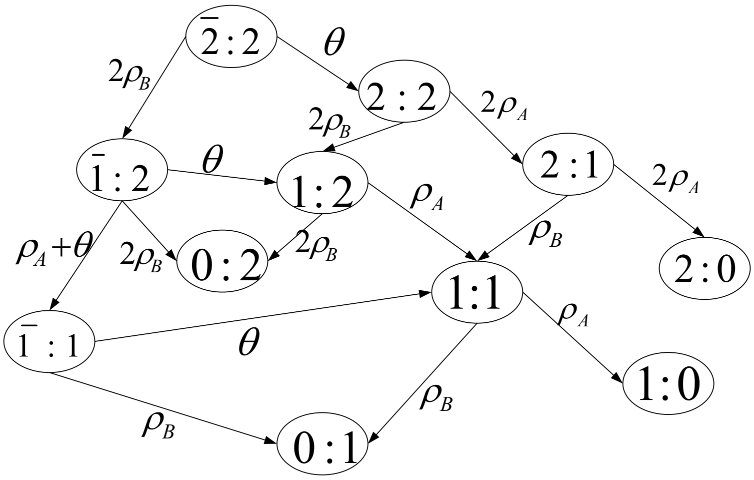

Figure 1: State transfer figure of two-versus-two stochastic duel with searching

$P_i(t)$is the probability of the state i ($i=\bar{2}:2, 2:2, \bar{1}:2, 1:2, \bar{1}:1, 0:2, 0:2, 0:1, 2:1, 1:1, 2:0, 1:0$) when the time is t. From all the assumptions above we conclude that the stochastic duel is a Markov process which is continuous in time and discrete in states. After analyzing the state transfer figure, we get the equation groups as follows:

$$P_{\bar{2}:2}(t+\Delta t) = P_{\bar{2}:2}(t) - (\lambda_{\bar{2}:2\to 2:2} + \lambda_{\bar{2}:2\to\bar{1}:2})\Delta t P_{\bar{2}:2}(t) + o(\Delta t)$$

$$P_{2:2}(t+\Delta t) = P_{2:2}(t) + \lambda_{\bar{2}:2\to 2:2}\Delta t P_{\bar{2}:2}(t) - (\lambda_{2:2\to 2:1} + \lambda_{2:2\to 1:2})\Delta t P_{2:2}(t) + o(\Delta t)$$

$$P_{\bar{1}:2}(t+\Delta t) = P_{\bar{1}:2}(t) + \lambda_{\bar{2}:2\to\bar{1}:2}\Delta t P_{\bar{2}:2}(t) - (\lambda_{\bar{1}:2\to\bar{1}:1} + \lambda_{\bar{1}:2\to 1:2} + \lambda_{\bar{1}:2\to 0:2})\Delta t P_{\bar{1}:2}(t) + o(\Delta t)$$

$$\begin{aligned} P_{1:2}(t+\Delta t) = P_{1:2}(t) &+ \lambda_{2:2\to 1:2}\Delta t P_{2:2}(t) + \lambda_{\bar{1}:2\to 1:2}\Delta t P_{\bar{1}:2}(t) \\ &- (\lambda_{1:2\to 1:1} + \lambda_{1:2\to 0:2})\Delta t P_{1:2}(t) + o(\Delta t) \end{aligned}$$

$$P_{2:1}(t+\Delta t) = P_{2:1}(t) + \lambda_{2:2\to 2:1}\Delta t P_{2:2}(t) - (\lambda_{2:1\to 1:1} + \lambda_{2:1\to 2:0})\Delta t P_{2:1}(t) + o(\Delta t)$$

$$\begin{aligned} P_{1:1}(t+\Delta t) = P_{1:1}(t) &+ \lambda_{2:1\to 1:1}\Delta t P_{2:1}(t) + \lambda_{1:2\to 1:1}\Delta t P_{1:2}(t) + \lambda_{\bar{1}:1\to 1:1}\Delta t P_{\bar{1}:1}(t) \\ &-(\lambda_{1:1\to 0:1} + \lambda_{1:1\to 1:0})\Delta t P_{1:1}(t) + o(\Delta t) \end{aligned} \quad (4)$$

$$P_{\bar{1}:1}(t+\Delta t)=P_{\bar{1}:1}(t)+\lambda_{\bar{1}:2\to\bar{1}:1}\Delta t P_{\bar{1}:2}(t)-(\lambda_{\bar{1}:1\to 1:1}+\lambda_{\bar{1}:1\to 0:1})\Delta t P_{\bar{1}:1}(t)+o(\Delta t)$$

$$P_{0:1}(t+\Delta t)=P_{0:1}(t)+\lambda_{\bar{1}:1\to 0:1}\Delta t P_{\bar{1}:1}(t)+\lambda_{1:1\to 0:1}\Delta t P_{1:1}(t)+o(\Delta t)$$

$$P_{0:2}(t+\Delta t)=P_{0:2}(t)+\lambda_{\bar{1}:2\to 0:2}\Delta t P_{\bar{1}:2}(t)+\lambda_{1:2\to 0:2}\Delta t P_{1:2}(t)+o(\Delta t)$$

$$P_{2:0}(t+\Delta t)=P_{2:0}(t)+\lambda_{2:1\to 2:0}\Delta t P_{2:1}(t)+o(\Delta t)$$

$$P_{1:0}(t+\Delta t)=P_{1:0}(t)+\lambda_{1:1\to 1:0}\Delta t P_{1:1}(t)+o(\Delta t)$$

Do transpositions to the equation groups above and divided by Δt while $\Delta t\to 0$, the equation groups above will become:

$$dP_{\bar{2}:2}(t)/dt=-(\lambda_{\bar{2}:2\to 2:2}+\lambda_{\bar{2}:2\to\bar{1}:2})P_{\bar{2}:2}(t)=-(\theta+2\rho_B)P_{\bar{2}:2}(t)$$

$$dP_{2:2}(t)/dt=\theta P_{\bar{2}:2}(t)-(2\rho_A+2\rho_B)P_{2:2}(t)$$

$$dP_{\bar{1}:2}(t)/dt=2\rho_B P_{\bar{2}:2}(t)-(2\theta+\rho_A+2\rho_B)P_{\bar{1}:2}(t)$$

$$dP_{1:2}(t)/dt=\theta P_{\bar{1}:2}(t)+2\rho_B P_{2:2}(t)-(\rho_A+2\rho_B)P_{1:2}(t)$$

$$dP_{2:1}(t)/dt=2\rho_A P_{2:2}(t)-(2\rho_A+\rho_B)P_{2:1}(t)$$

$$dP_{1:1}(t)/dt=\theta P_{\bar{1}:1}(t)+\rho_A P_{1:2}(t)+\rho_B P_{2:1}(t)-(\rho_A+\rho_B)P_{1:1}(t) \tag{5}$$

$$dP_{\bar{1}:1}(t)/dt=(\rho_A+\theta)P_{\bar{1}:2}(t)-(\theta+\rho_B)P_{\bar{1}:1}(t)$$

$$dP_{0:2}(t)/dt=2\rho_B\left[P_{\bar{1}:2}(t)+P_{1:2}(t)\right]$$

$$dP_{0:1}(t)/dt=\rho_B\left[P_{\bar{1}:1}(t)+P_{1:1}(t)\right]$$

$$dP_{2:0}(t)/dt=2\rho_A P_{2:1}(t)$$

$$dP_{1:0}(t)/dt=\rho_A P_{1:1}(t)$$

One property of Laplace transformation is:

$$L[dP_i(s)/ds] = sL[P_i(t)] - P_i(0) \tag{6}$$

$$(i = \overline{2}:2, 2:2, \overline{1}:2, 1:2, \overline{1}:1, 0:2, 0:2, 0:1, 2:1, 1:1, 2:0, 1:0)$$

Use Laplace transformation to the left and right sides of the equation groups on initial condition that

$$P_{\overline{2}:2}(0) = 1, P_i(0) = 0 (i = 2:2, \overline{1}:2, 1:2, \overline{1}:1, 0:2, 0:2, 0:1, 2:1, 1:1, 2:0, 1:0)$$

We use the notation L_i instead of $L[P_i(s)]$, (i = $\overline{2}$:2,2:2,$\overline{1}$:2,1:2,$\overline{1}$
:1,0:2,0:2,0:1,2:1,1:1,2:0,1:0) and will get the equation groups:

$$\begin{array}{ll}
sL_{\overline{2}:2} - 1 = -(\theta + 2\rho_B)L_{\overline{2}:2} & sL_{2:2} = \theta L_{\overline{2}:2} - (2\rho_A + 2\rho_B)L_{2:2} \\
sL_{\overline{1}:2} = 2\rho_B L_{\overline{2}:2} - (2\theta + \rho_A + 2\rho_B)L_{\overline{1}:2} & sL_{1:2} = \theta L_{\overline{1}:2} + 2\rho_B L_{2:2} - (\rho_A + 2\rho_B)L_{1:2} \\
sL_{2:1} = 2\rho_A L_{2:2} - (2\rho_A + \rho_B)L_{2:1} & sL_{1:1} = \theta L_{\overline{1}:1} + \rho_A L_{1:2} + \rho_B L_{2:1} - (\rho_A + \rho_B)L_{1:1} \\
sL_{\overline{1}:1} = (\rho_A + \theta)L_{\overline{1}:2} - (\theta + \rho_B)L_{\overline{1}:1} & sL_{0:2} = 2\rho_B(L_{\overline{1}:2} + L_{1:2}) \\
sL_{0:1} = \rho_B(L_{\overline{1}:1} + L_{1:1}) \quad sL_{2:0} = 2\rho_A L_{2:1} & sL_{1:0} = \rho_A L_{1:1}
\end{array} \tag{7}$$

So we get the solutions to the equation groups as follows:

$$L_{\overline{2}:2} = \frac{1}{(s+\theta+2\rho_B)} \qquad L_{2:2} = \frac{1}{(s+\theta+2\rho_B)(s+2\rho_A+2\rho_B)}$$
$$L_{\overline{1}:2} = \frac{2\rho_B}{(s+\theta+2\rho_B)(s+2\theta+\rho_A+2\rho_B)}$$

$$L_{1:2} = \frac{2\theta\rho_B}{(s+\theta+2\rho_B)(s+2\theta+\rho_A+2\rho_B)(s+\rho_A+2\rho_B)} + \frac{2\theta\rho_B}{(s+\theta+2\rho_B)(s+2\rho_A+2\rho_B)(s+\rho_A+2\rho_B)}$$

$$L_{2:1} = \frac{2\theta\rho_A}{(s+\theta+2\rho_B)(s+2\rho_A+2\rho_B)(s+2\rho_A+\rho_B)}$$
$$L_{\overline{1}:1} = \frac{2(\rho_A+\theta)\rho_B}{(s+\theta+2\rho_B)(s+2\theta+\rho_A+2\rho_B)(s+\theta+\rho_B)}$$

$$\begin{aligned}
L_{1:1} = & \frac{2\theta(\rho_A+\theta)\rho_B}{(s+\theta+2\rho_B)(s+2\theta+\rho_A+2\rho_B)(s+\theta+\rho_B)(s+\rho_A+\rho_B)} \\
& + \frac{2\theta\rho_A\rho_B}{(s+\theta+2\rho_B)(s+2\theta+\rho_A+2\rho_B)(s+\rho_A+2\rho_B)(s+\rho_A+\rho_B)} \\
& \frac{2\theta\rho_A\rho_B}{(s+\theta+2\rho_B)(s+2\rho_A+2\rho_B)(s+\rho_A+2\rho_B)(s+\rho_A+\rho_B)} \\
& + \frac{2\theta\rho_A\rho_B}{(s+\theta+2\rho_B)(s+2\rho_A+2\rho_B)(s+2\rho_A+\rho_B)(s+\rho_A+\rho_B)}
\end{aligned}$$

$$\begin{aligned}
L_{0:2} = \frac{1}{s} \Bigg[& \frac{4\rho_B^2}{(s+\theta+2\rho_B)(s+2\theta+\rho_A+2\rho_B)} + \frac{4\theta\rho_B^2}{(s+\theta+2\rho_B)(s+2\theta+\rho_A+2\rho_B)(s+\rho_A+2\rho_B)} \\
& + \frac{4\theta\rho_B^2}{(s+\theta+2\rho_B)(s+2\rho_A+2\rho_B)(s+\rho_A+2\rho_B)} \Bigg]
\end{aligned}$$

$$L_{0:1} = \frac{1}{s}\left[\frac{2(\rho_A+\theta)\rho_B^2}{(s+\theta+2\rho_B)(s+2\theta+\rho_A+2\rho_B)(s+\theta+\rho_B)} + \rho_B L_{1:1}\right]$$

$$L_{2:0} = \frac{4\theta\rho_A^2}{s(s+\theta+2\rho_B)(s+2\rho_A+2\rho_B)(s+2\rho_A+\rho_B)} \qquad L_{1:0} = \frac{\rho_A}{s}L_{1:1}$$

Mark the notations $P_A(2:2)$ and $P_B(2:2)$ as the winning probabilities of the two contestants, A and B. With the property of Laplace transformation we will get the winning probabilities of the two contestants, A and B as follows:

$$\begin{aligned} P_A(2:2) = & \lim_{t\to\infty}[P_{2:0}(t)+P_{1:0}(t)] = \lim_{s\to 0}(sL_{2:0}+sL_{1:0}) \\ & = \frac{2\theta\rho_A^2}{(\theta+2\rho_B)(\rho_A+\rho_B)(2\rho_A+\rho_B)} + \frac{2\theta\rho_A(\rho_A+\theta)\rho_B}{(\theta+2\rho_B)(\theta+\rho_B)(2\theta+\rho_A+2\rho_B)(\rho_A+\rho_B)} \\ & + \frac{2\theta\rho_A^2\rho_B}{(\theta+2\rho_B)(\rho_A+2\rho_B)(2\theta+\rho_A+2\rho_B)(\rho_A+\rho_B)} \\ & + \frac{3\theta\rho_A^2\rho_B}{(\theta+2\rho_B)(\rho_A+\rho_B)(\rho_A+2\rho_B)(2\rho_A+\rho_B)} \end{aligned} \tag{8}$$

$$\begin{aligned} P_B(2:2) = & \lim_{t\to\infty}[P_{0:2}(t)+P_{0:1}(t)] = \lim_{s\to 0}(sL_{0:2}+sL_{0:1}) \\ & = \frac{4\theta\rho_B^2}{(\theta+2\rho_B)(2\theta+\rho_A+2\rho_B)(\rho_A+2\rho_B)} \\ & + \frac{2\theta\rho_B^2}{(\theta+2\rho_B)(\rho_A+\rho_B)(\rho_A+2\rho_B)} + \frac{4\rho_B^2+2(\rho_A^+\theta)\rho_B^2}{(\theta+2\rho_B)(2\theta+\rho_A+2\rho_B)(\theta+\rho_B)} \\ & + \frac{2\theta(\rho_A^+\theta)\rho_B^2}{(\theta+2\rho_B)(\theta+\rho_B)(2\theta+\rho_A+2\rho_B)(\rho_A+\rho_B)} \\ & + \frac{2\theta\rho_A\rho_{B2}}{(\theta+2\rho_B)(\rho_A+2\rho_B)(2\theta+\rho_A+2\rho_B)(\rho_A+\rho_B)} \\ & + \frac{3\theta\rho_A\rho_B^2}{(\theta+2\rho_B)(\rho_A+\rho_B)(\rho_A+2\rho_B)(2\rho_A+\rho_B)} \end{aligned} \tag{9}$$

Also we will get the conclusion:

$$P_A(2:2)+P_B(2:2) = 1 \tag{10}$$

4 Simulation & analysis

Let us assume that in one combat there are two contestants, A and B, each of them has two units. There are two conditions:first condition is $\rho_B < \rho_A < 60$and $\rho_A = \rho_B + 5$, second condition is $\rho_A = \rho_B < 60$, the unit of which are 1. Also we assume that the search rate of A, $\theta = 100$, the unit of which is 1 and the simulation result is illustrated in Fig.2.

From the Fig.2 we can conclude that the winning probability of B which is stealthily, is higher than A. The winning probability of B becomes higher when ρ_B grows bigger which is illustrated in the right part of Fig.2.

5 Conclusions

In this paper the winning probability formulas of two-versus-two stochastic duel with searching are deduced using state transfer figure and the property of Laplace transformation. We can also get the formulas of three-versus-two, five-versus-three, six-versus-four etc., which are typical in real life. The results of the winning probabilities formulas are exact probabilistic solutions, which contain no time parameters. And the formulas can be used to quantificationally evaluate operation efficiency of more realistic small-to-moderate-size firefight models. That is why the stochastic duel models are so popular.

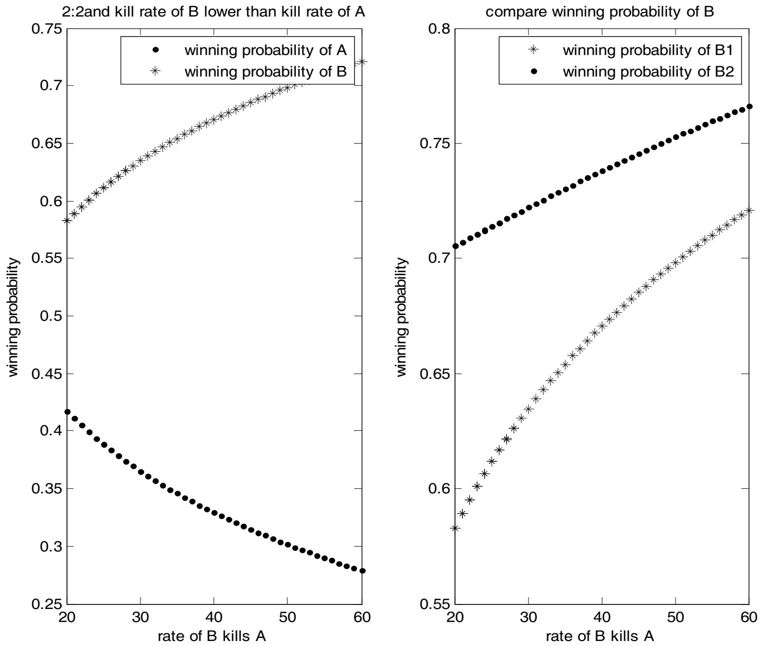

Figure 2: Simulation result of winning probability

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