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On Minimizing the Numbers of ADMs– Better Bounds for an Algorithm with Preprocessing*

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Abstract Minimizing the number of electronic switches in optical networks is a main research topic in recent studies. In such networks we assign colors to a given set of lightpaths. Thus the lightpaths are partitioned into cycles and paths, and the switching cost is minimized when the number of paths is minimized. The problem of minimizing the switching cost is NP-hard. A basic approximation algorithm for this problem eliminates cycles of size at most l and has a performance guarantee of $OPT+\frac{1}{2}N(1+\varepsilon)$, where OPT is the cost of an optimal solution, N is the number of lightpaths and $0 \leq \varepsilon \leq \frac{1}{l+2}$, for any given odd l. Shalom improved the analysis of this algorithm and prove that $\frac{1}{2l+3} \leq \varepsilon \leq \frac{1}{\frac{3}{2}(l+2)}$. In this paper, we further reduce the gap between the lower bound and the upper bound of ε. We show that a better upper bound of ε by constructing a greater matching, i.e., $\varepsilon \leq \frac{1}{\frac{5}{3}(l+2)}$.

Keywords Wavelength Assignment; Wavelength Division Multiplexing(WDM); Optical Networks; Add-Drop Multiplexer(ADM)

1 Introduction

1.1 Background

Given a WDM network $G=(V,E)$ comprising optical nodes and a set of fullduplex lightpaths $P=\{p_1,p_2,...,p_N\}$ of G, the wavelength assignment (WLA) task is to assign a wavelength to each lightpath p_i.

In the following discussion we also assume that each lightpath $p \in P$ is contained in a cycle of G. Each lightpath p uses two ADMs, one at each endpoint. Although only the downstream ADM function is needed at one end and only the upstream ADM function is needed at the other end, full ADMs will be installed on both nodes in order to complete the protection path around some ring. The full configuration would result in a number of SONET rings. It follows that if two adjacent lightpaths are assigned the same wavelength, then they can be used by the same SONET ring and the ADM in the common node can be shared by them. This would save the cost of one ADM. An ADM may be shared by at most two lightpaths. A more detailed technical explanation can be found in [1].

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Lightpaths sharing ADMs in a common endpoint can be thought as concatenated, so that they form longer paths or cycles. Each of these longer paths/cycles does not use any edge $e \in E$ twice, because otherwise they cannot use the same wavelength and this is a necessary condition to share ADMs.

1.2 Previous Work

Minimizing the number of electronic switches in optical networks is a main research topic in recent studies. The problem was introduced in [1] for ring topology. Approximation algorithm for ring topology with approximation ratio of $\frac{3}{2}$ was presented in [2], and was improved in [3,4] to $\frac{10}{7}+\varepsilon$ and $\frac{10}{7}$, respectively. For general topology [5] describe an algorithm with approximation ratio of $8/5$. The same problem was studied in [6] and an algorithm was presented that has a preprocessing phase where cycles of length at most l are included in the solution. The algorithm was shown to have a performance guarantee of

$$\text{OPT}+\frac{1}{2}N(1+\varepsilon), 0 \le \varepsilon \le \frac{1}{l+2} \tag{1}$$

where OPT is the cost of an optimal solution, N is the number of lightpaths, for any given odd l.

For $l = 1$ this implies algorithm without preprocessing, having a performance guarantee of $OPT + \frac{2}{3}N$. In [7] the algorithm is proven to have a performance guarantee of $\text{OPT}+\frac{3}{5}N$.

By suggesting a novel technique including a new combinatorial lemma, Shalom improve the analysis of this algorithm with preprocessing phase in [8], and prove that

$$\text{OPT}+\frac{1}{2}N(1+\varepsilon), \frac{1}{2l+3} \le \varepsilon \le \frac{1}{\frac{3}{2}(l+2)}. \tag{2}$$

1.3 Our contribution

We further reduce the gap between the lower bound and the upper bound of ε on the basis of [8]. We improve the analysis of the algorithm with preprocessing phase in [8], and prove a performance guarantee of

$$OPT+\frac{1}{2}N(1+\varepsilon), \frac{1}{2l+3} \le \varepsilon \le \frac{1}{\frac{5}{3}(l+2)}. \tag{3}$$

Our improving sheds more light on the structure and properties of the algorithm, by closely examining the structural relation between the solution found by the algorithm and an optimal solution, for any given instance of the problem. As the running time of the algorithm is exponential in l, our results imply an improvement in the analysis of the running time of the algorithm.

In Section 2 we describe the problem and give some preliminary results. Our contribution in further improving the analysis of the algorithm are presented in Section 3. In Section 4 we make the conclusion.

2 Problem Definition and Preliminary Results

2.1 Problem Definition

An instance α of the problem is a pair $\alpha = (G,P)$, where $G = (V,E)$ is an undirected graph and P is a set of simple paths in G. Given such an instance we define the following:

Definition 2.1 The paths $p, p^{'}$ ($p, p^{'} \in P$) are conflicting or overlapping if they have a common edge. This is denoted as $p \asymp p^{'}$. The graph of the relation $\asymp$ is called the conflict graph of (G,P).

Definition 2.2 A proper coloring (or wavelength assignment) of P is a function ω : $P \mapsto \mathbb{N}$, such that $\omega(p) \neq \omega(p^{'})$ whenever $p \asymp p^{'}$.

Definition 2.3 A valid chain (resp. cycle) of an instance α is a path (resp. cycle) formed by the concatenation of distinct paths $p_0, p_1, ..., p_{k-1} \in P$ that do not go over the same edge twice.

Definition 2.4 A solution S of an instance $\alpha = (G,P)$ is a set of chains and cycles of P such that each $p \in P$ appears in exactly one of these sets.

Definition 2.5 The shareability graph of an instance $\alpha = (G,P)$ is the edge-labelled multi-graph $\mathscr{G}_\alpha = (P, E_\alpha)$ such that there is an edge $e = (p,q)$ labelled u in E_α if and only if $p \not\asymp q$, and u is a common endpoint of p and q in G.

Example: Let $\alpha = (G,P)$ be the instance in Figure 2.1. Its shareability graph $\mathscr{G}_\alpha$ is the graph at the left of the Figure 2.2. In this instance $P = \{a,b,c,d,e,f\}$, and it constitutes the set of nodes of $\mathscr{G}_\alpha$. The edges together with their labels are

$E_\alpha = \{(b,c,u),(d,e,x),(a,c,w),(a,b,x),(e,d,x),(b,d,x),(d,f,v),(b,e,x)\}$.

Because b and c can be joined in their common endpoint u, etc. The corresponding conflict graph is in the middle of the figure 2.2. It has the same node set and the edge set is $\{(a,d),(c,d)\}$. The paths $c,d \in P$ are conflicting because they have a common edge, i.e.(w,v),etc.

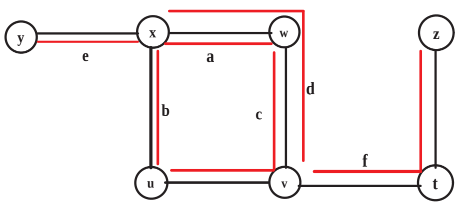


Figure.2.1 A sample input

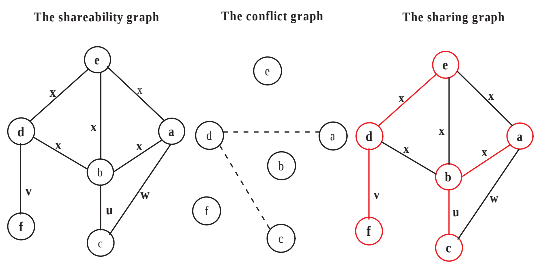


Figure.2.2 The shareability, conflict and sharing graphs

Note that the edges of the conflict graph are not in E_α, and for any node v of $\mathcal{G}_\alpha$, the set of labels of the edges adjacent to v is of size at most two.

Definition 2.6 The sharing graph of a solution S of an instance α is the following subgraph $\mathcal{G}_{\alpha,S} = (P, E_S)$ of $\mathcal{G}_\alpha$. Two lightpaths $p, q \in P$ are connected with an edge labelled u in E_S if and only if they are consecutive in a chain or cycle in the solution S, and their common endpoint is $u \in V$. We will shortly write $\mathcal{G}_S$.

Definition 2.7 The degree of a node p in $\mathcal{G}_S$ is the size of the set of labels of the edges adjacent to it, which is denoted as $d(p)$, for $d(p) = \{0,1,2\}$.

In our example, $S = \{(a,b,c),(e,d,f)\}$ is a solution with two chains. The sharing graph of this solution is shown at the right of the Figure 2.2. We define: $\forall i \in \{0,1,2\}, D_i(S) \stackrel{def}{=} \{p \in P | d(p) = i\}$, and $d_i(S) \stackrel{def}{=} |D_i(S)|$. Note that $d_0(S) + d_1(S) + d_2(S) = |P| = N$. An edge $(p,q) \in E_S$ with label u corresponds to a concatenation of two paths with the same color at their common endpoint u. Therefore these two endpoints can share an ADM operating at node u, thus saving one ADM. We conclude that every edge of E_S corresponds to a saving of one ADM. When no ADMs are shared, each path needs two ADMs, a total of $2N$ ADMs. Then the cost of a solution S is $2|P| - |E_S| = 2N - |E_S|$. The objective is to find a solution S such that $cost(S)$ is minimum, in other words $|E_S|$ is maximum.

2.2 Preliminary Results

Given a solution S, $d(p) \leq 2$ for every node $p \in P$, the connected components of $\mathcal{G}_S$ are either paths or cycles. We know that an isolated vertex is a special case of a path. Let $\mathcal{P}_{\mathcal{S}}$ be the set of the connected components of $\mathcal{G}_S$ that are paths. Let S^* be a solution with minimum cost. For any solution S we define $\varepsilon(S) \stackrel{def}{=} \frac{d_0(S) - d_2(S) - 2|\mathcal{P}_{S^*}|}{N}$.

Lemma 1 [8] For any solution S, $\cos t(S) = \cos t(S^*) + \frac{1}{2}N(1 + \varepsilon(S))$.

For the algorithm $PMM(l)$ presented in [2], the algorithm has a preprocessing phase which removes cycles of size at most l, where l is an odd number. Shalom improve the analysis in [8] and prove that $OPT + \frac{1}{2}N(1+\varepsilon), 0 \leq \varepsilon \leq \frac{1}{l+2}$. And also in [8] improve an better bound that $OPT + \frac{1}{2}N(1+\varepsilon), \frac{1}{2l+3} \leq \varepsilon \leq \frac{1}{\frac{3}{2}(l+2)}$.

3 Our Contribution

3.1 Algorithm $PMM(l)$

First we recall that the algorithm $PMM(l)$ presented in [2]. The algorithm has a preprocessing phase which removes cycles of size at most l, where l is an odd number. Then it proceeds to its processing phase(Function MM) which can be described as follows:

Begin with chains consisting of single nodes (which are always valid). At each iteration, combine a maximum number of pairs of chains to obtain longer chains. This is done by constructing an appropriate graph and computing a maximum matching on it. The algorithm ends when the maximum matching is empty, namely no two chains can be combined into a longer chain.

The procedure of the algorithm is described in [8] detailedly.

3.2 Correctness and Our Analysis

The correctness was proved in [8]. The previous proof in [2] is based on the existence of a matching M having certain size. This matching consists solely of edges of the connected components of $\mathscr{G}_{S^*}$. In our proof we show that using other edges of $\mathscr{G}_\alpha$ we can build a larger matching than the matching constructed in [8] which leads to a better upper bound. A lower bound was proved in [8]. In this subsection we build the improved matching and prove our better upper bound.

We begin by developing some results which will be used in our proof. Some related results have been proved in [8].

Definition 3.1 For every $X \subseteq P$, $OUT\,(C) \stackrel{def}{=} C\left(X, \overline{X}\right)$ is the cut of X in $\mathscr{G}_S$, namely the set of edges of $\mathscr{G}_S$ having exactly one endpoint in X.

Definition 3.2 The i-neighborhood $N_i(X)$ of X is the set of all the nodes having exactly i neighbors from X in $\mathscr{G}_S$, but are not in X. $N(X) \stackrel{def}{=} N_1(X)$.

Lemma 2 Let C be a cycle of $\mathscr{G}_{S^*}$, then $|OUT| \geq \frac{1}{3}(|C| + |D_0\,(S) \cap C| - |D_2\,(S) \cap C|)$.

The following lemma generalize the previous lemma to a set of cycles.

Lemma 3 Let $\mathscr{C}$ be a set of cycles of $\mathscr{G}_{S^*}$. Let $P_{\mathscr{C}} \stackrel{def}{=} \cup \mathscr{C}$ be the set of nodes of these cycles. Let $IN(\mathscr{C})$ be the set of edges of $\mathscr{G}_S$ connecting two cycles of $\mathscr{C}$. Then $|N(P_{\mathscr{C}})| \geq \frac{1}{3}|P_{\mathscr{C}}| + \frac{1}{3}|D_0(S) \cap P_{\mathscr{C}}| - \frac{1}{3}|D_2(S) \cap P_{\mathscr{C}}| - 2|IN(\mathscr{C})| - 2|N_2(P_{\mathscr{C}})|$.

Definition 3.3 The odd cycles graph $\mathscr{OG}_S = (\mathscr{OC}_S, \mathscr{O}\varepsilon_S)$ of a solution S is a graph in which each node corresponds to an odd cycle of $\mathscr{G}_{S^*}$ which does not intersect with P_0 and two nodes are connected with an edge if and only if there is an edge connecting the corresponding cycles in E_S.

Lemma 4 Let $\mathscr{X} \subseteq \mathscr{OC}_S$. Then, $|N(P_{\mathscr{X}})| \geq \frac{1}{3}|P_{\mathscr{X}}| - 2|IN(\mathscr{X})| - 2(d_2(S) - |P_0|)$.

Corollary 3.1 Let $\mathscr{I}$ be an independent set of $\mathscr{OG}_S$. Then, $|N(P_{\mathscr{I}})| \geq \frac{1}{3}|P_{\mathscr{I}}| - 2(d_2(S) - |P_0|)$.

Odd Distanced Nodes with Distinct Colors In this subsection we first recall that the definition of MODNDC (“maximum odd distanced nodes with distinct colors”) problem which is presented in [8]. The cycle version of the problem, (MODNDC -C) is defined as follows: The input and output is described in [8], where we have not repeated.

Measure: Our goal is to find V $(V = (v_0, v_1, ..., v_{k-1}))$ minimizing the number of nodes of C which are colored with colors from $\{c(v_0), c(v_1), ..., c(v_{k-1})\}$. Given a solution, we first set $c(v) = 0$ for all v such that $c(v) = \{c(v_0), c(v_1), ..., c(v_{k-1})\}$ and we count the number of nodes v with $c(v) = 0$. $B_c(V)$ is defined as the set of nodes colored c after this uncoloring, formally $B_c(V) \stackrel{def}{=} \{v \in C | c(v) = c\}$. $W(V) \stackrel{def}{=} B_0$ is the set of uncolored nodes. $B(V) \stackrel{def}{=} \uplus_{c>0} B_c$ is the set of colored nodes. Our target is to find a solution V such that $|W(V)|$ is maximized. Obviously, $C = B(V) \uplus W(V)$.

Definition 3.4 A cycle C is dedicated if it contains nodes colored with one color and possibly some uncolored nodes. Formally, $|\{c(v)|v \in C\} \setminus \{0\}| = 1$.

Lemma 5 Given an instance of the (MODNDC-C) problem, one of the following is true: (a) C is a dedicated even cycle, (b) There is a solution V with measure $|W(V)| \geq \left\lfloor \frac{n}{2} \right\rfloor$.

Proof: Let V be an optimal solution. We consider the following cases:

Case 1: $V = \emptyset$. The proof is similar to that of in [8].

Case 2: $V \neq \emptyset$. We want to show that $|W(V)| \geq \lfloor \frac{n}{2} \rfloor = \frac{|W(V)|}{2} + \frac{|B(V)|}{2}$ which is equivalent to $|B(V)| \leq |W(V)|$. For this purpose we will partition the set $B(V)$ into two disjoint sets X and Y, then prove $|X| + |Y| \leq |W(V)|$.

Let $V = \{v_0, v_1, ..., v_{k-1}\}$. Consider two consecutive nodes $v_i, v_j \in V$. Note that $i = j$ if $k = 1$, thus these nodes need not be distinct. Recall also that the clockwise distance $d(v_i, v_j)$ from v_i to v_j is odd.

Observe that if there are two colored nodes $x, y \in B(V)$ between these two nodes such that x is closer to v_i and that $d(v_i, x)$ and $d(x, y)$ are odd, then $c(x) = c(y)$. For, otherwise the set $V \uplus \{x, y\}$ is a better solution than V, a contradiction.

We use this observation to characterize the colored nodes of the solution, i.e. the nodes of $B(V)$. For the following discussion, we consult Figure 3.1. Let $x \in B(V)$ be the colored node which is closest to v_i when going clockwise from v_i to v_j and is at odd distance from v_i. Let $y \in B(V)$ be the colored node which is farthest from v_i when going from v_i to v_j and is at even distance from v_i. Note that y is the first node in $B(V)$ at odd distance from v_j when going counterclockwise from v_j to v_i. By these choices, all the colored nodes before x are at even distance from v_i and all the colored nodes after y are at odd distance from v_i. If y occurs before x then there are no colored nodes between x and y, or in other words, all the colored nodes are either before y or after x. This statement holds even if one or both of x, y do not exist. In all these cases we define $X_i = \emptyset$. If y occurs after x then by the observation in the previous paragraph $c(x) = c(y) = c$. Furthermore, for every colored node z between x and y, $c(z)$=c. In this case we define X_i be the set of all the colored nodes from x to y including x and y. Let also Y_i be the set of all other colored nodes between v_i and v_j. Let $X \stackrel{def}{=} \uplus_{i=0}^{k-1} X_i$ and $Y \stackrel{def}{=} \uplus_{i=0}^{k-1} Y_i$.

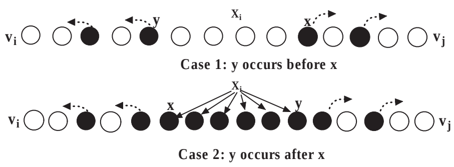


Figure.3.1. The nodes between two nodes of the solution

Let $V_j \subseteq W$ be the set of nodes having originally the same color as v_j. Note that Y_i has at least one node y which is at even distance from v_j. Similarly, X_i has at least one node x which is at odd distance from v_j. Therefore $V' = V \setminus \{v_j\} \cup \{y\} \cup \{x\}$ is a solution. If $|Y_i| + |X_i| > |V_j|$ then $|W(V')| > |W(V)|$, contradicting the fact that V is an optimal solution, hence $|Y_i| + |X_i| \leq |V_j|$. Summing up from $i = 0$ to $k - 1$ we have $|X| + |Y| \leq \uplus_{i=0}^{k-1} |V_j| \leq |W|$.

We conclude that $|X| + |Y| \leq |W(V)|$, then $|B(V)| \leq |W(V)|$. Clearly, we have $|W(V)| \geq \lfloor \frac{n}{2} \rfloor$.

The proof is completed.

Corollary 3.2 Given an instance of the (MODNDC-P) problem, which is the path version of MODNDC problems, there is always a solution V with measure $|W(V)| \geq \lfloor \frac{n}{2} \rfloor$.

Proof: In that case there are at least two nodes colored with two different colors, we connect the endpoints of P using an edge to constitute a cycle C. Then the proof is similar to the MODNDC-C problem which has an instance with n nodes and C is not a dedicated cycle. By the previous lemma, there is a solution of this instance with measure at least $\frac{n}{2}$. This solution satisfies the conditions of MODNDC-P problem too. Specially case, if none of the nodes are colored, then the empty set is a solution with measure n. If all the colored nodes have the same color, then any one of these nodes constitutes a solution with measure n. Then the result is proved.

An improved better upper bound

Theorem 1 Given a solution S of $PMM(l)$, $\varepsilon(S) \leq \frac{1}{\frac{5}{3}(l+2)}$.

Proof: Our proof constructs a different (and larger) matching M, and the constructing is similar to that in [8]. We partition the connected components of $\mathscr{G}_{S^*}$ as follows:

$\mathscr{I}$ is some maximum independent set of $\mathscr{O}\mathscr{G}_S$; $\mathscr{D} = \mathscr{O}\mathscr{C}_S \setminus \mathscr{I}$; $\mathscr{O}_d$ is the set of all odd cycles of $\mathscr{G}_{S^*}$ except those in $\mathscr{O}\mathscr{C}_S$, in other words, all the odd cycles of $\mathscr{G}_{S^*}$ which intersect with P_0, such that $|\mathscr{O}_d \cap P_0|$ is odd; $\mathscr{O}_e$ is the set of all odd cycles of $\mathscr{G}_{S^*}$ except those in $\mathscr{O}\mathscr{C}_S$, in other words, all the odd cycles of $\mathscr{G}_{S^*}$ which intersect with P_0,such that $|\mathscr{O}_e \cap P_0|$ is even; ε is the set of even cycles of $\mathscr{G}_{S^*}$; P_{S^*} is the set of maximal paths of $\mathscr{G}_{S^*}$.

Note that each cycle in $\mathscr{O}\mathscr{C}_S = \mathscr{I} \uplus \mathscr{D}$ has at least $l+2$ nodes. We further partition these sets as follows:

$\mathscr{I} = \mathscr{I}_1 \uplus \mathscr{I}_2 \uplus \mathscr{I}_D$; $\mathscr{D} = \mathscr{D}_1 \uplus \mathscr{D}_2$; $\mathscr{O}_d = \mathscr{O}_{d1} \uplus \mathscr{O}_{d2}$; ; $\mathscr{O}_e = \mathscr{O}_{e1} \uplus \mathscr{O}_{e2}$; $\varepsilon = \varepsilon_D \uplus \varepsilon_2$.

Initially $\mathscr{I}_D = \mathscr{I}_2 = \mathscr{D}_2 = \mathscr{O}_{d2} = \mathscr{O}_{e2} = \varepsilon_2 = \emptyset$, thus $\mathscr{I}_1 = \mathscr{I}$, $\mathscr{D}_1 = \mathscr{D}$, $\mathscr{O}_{d1} = \mathscr{O}_d$, $\mathscr{O}_{e1} = \mathscr{O}_e$, $\varepsilon_D = \varepsilon$ and M is a empty matching.

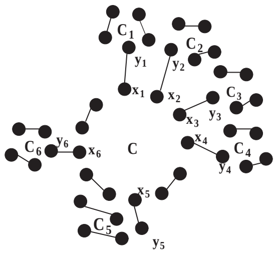
Phase 1-Coloring: The processing of this phase is similar to that of the preliminary results in [8]. At this point the following two invariants are obviously true.

CN1: All the nodes in the cycles of $\varepsilon_2 \cup \mathscr{D}_2 \cup \mathscr{I}_2$ are covered by M.

CN2: There is a one to one correspondence between the set of colors and the set of cycles in $\mathscr{I}_1 \cup \mathscr{I}_D$.

Phase 2-Uncoloring by MODNDC-C of even cycles: As long as there is an even cycle C in ε_D, admiting a solution with measure at least $\frac{n}{2}$ to the MODNDC-C problem, do the following processing which is described in Figure 3.2.

Pick an optimal solution of the MODNDC problem for C with the current colors. Let $x_1, x_2, ...x_k$ be the nodes of the solution. Clearly, k is even. Let y_i be the neighbor node x_i which gave it its color in Phase 1. As the colors of each x_i are distinct, the nodes y_i belong to distinct odd cycles $C_i \in \mathscr{I}$. Let p_i be the path on C from x_i to x_{i+1} excluding x_i and x_{i+1}, if the path p_i exists, then it is a path of odd length. As such, these paths admit a perfect matching. The induced subgraph resulting from the removal of y_i from C_i is a path with an even number of nodes, and admit a perfect matching too. Add these matching and the edges $\{(x_i, y_i)|i \leq k\}$ to M. Now M cover perfectly the cycles $C, C_1, C_2, ..., C_k$. In particular if $k = 0$ then M covers perfectly C. Uncolor all the nodes with colors $c(x_1), c(x_2), ..., c(x_k)$ in $\mathscr{G}_\alpha$, then



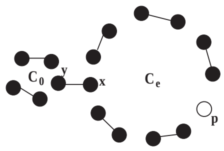
$$\mathscr{I}_1 \leftarrow \mathscr{I}_1 \setminus \{C_1, C_2, \ldots, C_k\};$$
$$\mathscr{I}_2 \leftarrow \mathscr{I}_2 \cup \{C_1, C_2, \ldots, C_k\};$$
$$\varepsilon_D \leftarrow \varepsilon_D \setminus \{C\}; \;\; \varepsilon_2 \leftarrow \varepsilon_2 \cup \{C\}.$$

Figure.3.2. Matching by MODNDC-C of even cycles

Note, ε_D may contain only dedicated cycles C with at least $\left\lfloor \frac{|C|}{2} \right\rfloor$ colored nodes.

Phase 3-Uncoloring by preprocessed dedicated even cycles: For every even cycle C_e such that $C_e \bigcap P_0 \neq \emptyset$ do the following processing which is described in Figure 3.3.

Pick arbitrarily a node $p \in C_e \bigcap P_0$. There are at least $\left\lfloor \frac{|C_e|}{2} \right\rfloor$ colored nodes in C_e, therefore there is at least one colored node x at odd distance from p. The node has a neighbor y in a cycle $C_0 \in \mathscr{I}_1$. $C_0 \setminus \{y\}$ is an odd path. $C_e \setminus \{p, x\}$ consists of two odd paths. They admit perfect matching. Add these matching and (x, y) to M. Uncolor all the nodes with colors $c(x)$ in $\mathscr{G}_\alpha$, then



$$I_1 \leftarrow I_1 \setminus \{C_0\}; \; I_2 \leftarrow I_2 \cup \{C_0\};$$
$$\varepsilon_D \leftarrow \varepsilon_D \setminus \{C_e\}; \; \varepsilon_2 \leftarrow \varepsilon_2 \cup \{C_e\}$$

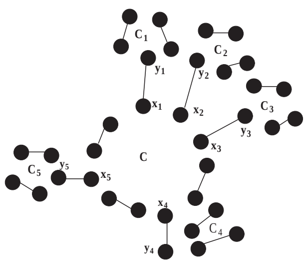
Figure.3.3. Matching by using preprocessed dedicated even cycles

Invariants CN1, CN2 hold, and the following invariant also holds.

CN3: ε_D contains only dedicated even cycles which do not intersect with P_0.

Phase 4-Uncoloring by MODNDC of odd cycles: For every odd cycle $C \in \mathscr{D}_1$ we do the following:

Pick an optimal solution of the MODNDC-C problem for C with the current colors. Let $x_1, x_2, \ldots, x_k$ be the nodes of the solution. Clearly, k is either zero or odd. If $k > 0$ build a perfect matching as in phase 2, consult the Figure 3.4. Then

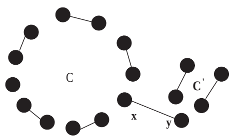


$$\mathscr{I}_1 \leftarrow \mathscr{I}_1 \setminus \{C_1, C_2, \ldots C_k\};$$
$$\mathscr{I}_2 \leftarrow \mathscr{I}_2 \cup \{C_1, C_2, \ldots C_k\};$$
$$\mathscr{D}_1 \leftarrow \mathscr{D}_1 \setminus \{C\}; \; \mathscr{D}_2 \leftarrow \mathscr{D}_2 \cup \{C\}$$

Figure.3.4. Matching by MODNDC of odd cycles

Phase 5-Match odd cycles in $\mathscr{D}_1$: Find a maximum matching of $\mathscr{D}_1$(consult Figure 3.5). For each pair of cycles $C, C^{'}$ in this matching do the following :

Pick arbitrarily an edge joining these two cycles in $\mathscr{G}_S$, add it to M. The remaining parts of C and $C^{'}$ are paths with an even number of nodes each of which admit a perfect matching. Add these perfect matching to M.

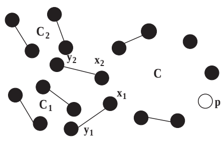


$\mathscr{D}_1 \leftarrow \mathscr{D}_1 \backslash \{C, C^{'}\}$; $\mathscr{D}_2 \leftarrow \mathscr{D}_2 \cup \{C, C^{'}\}$

Figure.3.5. Reducing $\mathscr{D}_1$

Phase 6-Cover preprocessed odd cycles in $\mathscr{O}_d$: For every odd cycle $C \in \mathscr{O}_d$ such that $|C \cap P_0|$ is odd, do the following processing (consult figure 3.6):

Pick arbitrarily a node $p \in C \cap P_0$. There are at least $\left\lfloor \frac{|C|}{2} \right\rfloor$ colored nodes in C, therefore there is at least one colored node x_1 at odd distance from p. And also having one colored node x_2 at odd distance from p in counterclockwise, because $C \in \mathscr{O}_d$ is odd cycles. The node x_1 has a neighbor y_1 in a cycle $C_1 \in \mathscr{I}_1$, and the node x_2 also has a neighbor y_2 in a cycle $C_2 \in \mathscr{I}_1$. $C_1 \setminus \{y_1\}$ and $C_2 \setminus \{y_2\}$ are paths with even number of nodes. $C \setminus \{x_1, x_2, p\}$ consists of two odd paths which admit perfect matchings. Add these matching and $\{(x_1, y_1), (x_2, y_2)\}$ to M.

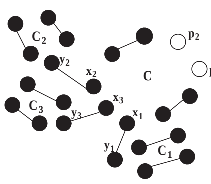


$\mathscr{O}_{d1} \leftarrow \mathscr{O}_{d1} \backslash \{C\}$; $\mathscr{O}_{d2} \leftarrow \mathscr{O}_{d2} \cup \{C\}$

Figure.3.6. Matching by MODNDC of odd cycles in $\mathscr{O}_d$

Phase 7-Cover Preprocessed odd cycles in $\mathscr{O}_e$: For every odd cycle $C \in \mathscr{O}_e$ such that $|C \cap P_0|$ is even, do the following processing (consult Figure 3.7):

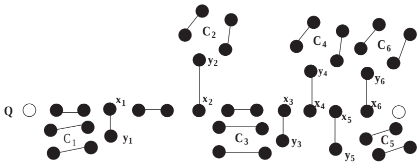
Pick arbitrarily two node $p_1 \in C \cap P_0$, $p_2 \in C \cap P_0$. There are at least $\left\lfloor \frac{|C|}{2} \right\rfloor$ colored nodes in C, then there is at least one colored node x_1 at odd distance from p_1, and also having a colored node x_2 at odd distance from p_2 in counterclockwise. The node x_1 has a neighbor y_1 in a cycle $C_1 \in \mathscr{I}_1$, and the colored node x_2 has a neighbor y_2 in a cycle $C_2 \in \mathscr{I}_1$. similarly, there is one colored node x_3 is at odd distance from x_1, and it also has a neighbor y_3 in $C_3 \in \mathscr{I}_1.C_1 \setminus \{y_1\}$, $C_2 \setminus \{y_2\}$ and $C_3 \setminus \{y_3\}$ are paths with even number of nodes. $C \setminus \{x_1, x_2, x_3, p_1, p_2\}$ consists of two paths with odd length. They admit perfect matchings. Add these matching and $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$ to M.



$\mathscr{O}_{e1} \leftarrow \mathscr{O}_{e1} \backslash \{C\}$; $\mathscr{O}_{e2} \leftarrow \mathscr{O}_{e2} \cup \{C\}$

Figure.3.7. Cover Preprocessed odd cycles in $\mathscr{O}_e$

Phase 8-Uncoloring by MODNDC-P: For every path $Q \in P_{S^*}$, do the following processing which is depicted in Figure 3.8. Pick an optimal solution of the MODNDC-P problem for Q with the current colors. Let $x_1, x_2, ..., x_k$ be the nodes of the solution. As the colors of the x_i are distinct, the neighbor nodes y_i of x_i belong to distinct odd cycles C_i. Let p_i be the path on C from x_i to x_{i+1} excluding x_i and x_{i+1}. If the path exists, then this is a path with odd length and admits a perfect matching. Removing y_i from C_i can form a path with odd length, which admits a perfect matching. Add these matching and the edges $\{(x_i, y_i)|i \le k\}$ to M. Now M perfectly covers the cycles $C_1, C_2, ..., C_k$. Uncolor all the nodes with colors $c(x_1), c(x_2), ..., c(x_k)$ in $\mathscr{G}_\alpha$, then,



$$\mathscr{I}_1 \leftarrow \mathscr{I}_1 \backslash \{C_1, C_2, ..., C_k\}$$
$$\mathscr{I}_2 \leftarrow \mathscr{I}_2 \cup \{C_1, C_2, ..., C_k\}$$

Figure.3.8. Matching by MODNDC-P

The remaining paths at both ends of Q may or may not admits a perfect matching. We add a maximum matching of each of them to M. We remain with at most two uncovered nodes of Q.

Phase 9-Cover ε_D: We know that ε_D contains only dedicated even cycles. Thus, for every cycle C_e in ε_D find a odd cycle $C_0 \in \mathscr{I}$ corresponding to the unique color of its colored nodes. C_e admits a perfect matching. Add this matching to M and then $\mathscr{I}_1 \leftarrow \mathscr{I}_1 \setminus \{C_0\}$; $\mathscr{I}_D \leftarrow \mathscr{I}_D \cup \{C_0\}$.

CN4: M covers the nodes of the cycles in ε_D.

Phase 10-Partly cover $\mathscr{I}_1 \cup \mathscr{I}_D \cup \mathscr{D}_1$: For every (odd) cycle $C \in \mathscr{I}_1 \cup \mathscr{I}_D \cup \mathscr{D}_1$, pick a node arbitrarily. The remaining nodes of C form a path with an even number of nodes, and admit a perfect matching. Add this matching to M.

At this point the construction of M is completed. The invariant CN1,CN2,CN3,and CN4 hold. In the sequel we will calculate an better upper bound of $\varepsilon(S)$.

By the construction we have $d_0(M) \le |\mathscr{I}_1| + |\mathscr{I}_D| + |\mathscr{D}_1| + 2|P_{S^*}|$. We get $d_0(S) \le d_0(M) + |P_0|$, in [8]. Then, $d_0(S) \le |\mathscr{I}_1| + |\mathscr{I}_D| + |\mathscr{D}_1| + 2|P_{S^*}| + |P_0|$

$$d_0(S) - d_2(S) - 2|P_{S^*}| \le |\mathscr{I}_1| + |\mathscr{I}_D| + |\mathscr{D}_1| - (d_2(S) - |P_0|). \tag{4}$$

Each dedicated cycle in ε_D has its nodes colored with one color. Then the number of colors used in all the cycles of ε_D is at most $|\varepsilon_D|$, and these colors have a one-to-one correspondence with the cycles of $\mathscr{I}_D$. Therefore

$$|\mathscr{I}_D| \le |\varepsilon_D|. \tag{5}$$

In [8], we also get

$$|\mathscr{D}_1| \le |\mathscr{I}_2|. \tag{6}$$

Combine (5) and (6), multiply both sides by $\frac{2}{3}(l+2)$, then

$$\frac{2}{3}(l+2)(|\mathscr{I}_D| + |\mathscr{D}_1|) \le \frac{2}{3}(l+2)(|\varepsilon_D| + |\mathscr{I}_2|) \le (l+1)|\varepsilon_D| + (l+2)|\mathscr{I}_2 \le |P_{\varepsilon_D}| + |P_{\mathscr{I}_2|}|. \tag{7}$$

Similarly,

$$(l+2)(|\mathscr{D}_1|+|\mathscr{I}_1|+|\mathscr{I}_D|) \leq |P_{\mathscr{D}_1}|+|P_{\mathscr{I}_1}|+|P_{\mathscr{I}_D}|. \tag{8}$$

For a component (cycle or chain) C_i of $\mathscr{G}_{S^*}$, let col_i be the number of the colored nodes in it, and let $uncol_i$ be the number of uncolored nodes in it. The nodes of $N(P_{\mathscr{I}_1})$ are all colored and they are in $P_{\mathscr{D}_2} \cup P_{\mathscr{O}_{d2}} \cup P_{\mathscr{O}_{e2}} \cup P_{\varepsilon_2} \cup P_{S^*}$, and there are no dedicated even cycles in $\mathscr{D}_2 \cup \mathscr{O}_{d2} \cup \mathscr{O}_{e2} \cup \varepsilon_2 \cup P_{S^*}$, therefore

$$|N(P_{\mathscr{I}_1})| \leq \sum_{C_i \in \mathscr{D}_2 \cup \mathscr{O}_{d2} \cup \mathscr{O}_{e2} \cup \varepsilon_2 \cup P_{S^*}} col_i. \tag{9}$$

On the MODNDC-C problems for each component C_i we have proved that $|B(V)| \leq |W(V)|$, which is equivalent to $col_i \leq uncol_i$. Then

$$|N(P_{\mathscr{I}_1})| \leq \sum_{C_i \in \mathscr{D}_2 \cup \mathscr{O}_{d2} \cup \mathscr{O}_{e2} \cup \varepsilon_2 \cup P_{S^*}} col_i \leq \sum_{C_i \in \mathscr{D}_2 \cup \mathscr{O}_{d2} \cup \mathscr{O}_{e2} \cup \varepsilon_2 \cup P_{S^*}} uncol_i. \tag{10}$$

Combining (9) and (10) and substituting $|C_i| = col_i + uncol_i$, we obtain

$$2|N(P_{\mathscr{I}_1})| \leq \sum_{C_i \in \mathscr{D}_2 \cup \mathscr{O}_{d2} \cup \mathscr{O}_{e2} \cup \varepsilon_2 \cup P_{S^*}} |C_i|.$$

Equivalently,

$$2|N(P_{\mathscr{I}_1})| \leq |P_{\mathscr{D}_2}|+|P_{\mathscr{O}_{d2}}|+|P_{\mathscr{O}_{e2}}|+|P_{\varepsilon_2}|+|\mathscr{P}_{P_{S^*}}|.$$

By Corollary 3.1, $|N(P_{\mathscr{I}_1})| \geq \frac{1}{3}|P_{\mathscr{I}_1}| - 2(d_2(S)-|P_0|)$, we have

$$2|N(P_{\mathscr{I}_1})| \geq \frac{2}{3}|P_{\mathscr{I}_1}| - 4(d_2(S)-|P_0|) \geq \frac{2}{3}(l+2)|\mathscr{I}_1| - 4(d_2(S)-|P_0|).$$

Combining these two inequality above, we get

$$\frac{2}{3}(l+2)|\mathscr{I}_1| - 4(d_2(S)-|P_0|) \leq |P_{\mathscr{D}_2}|+|P_{\mathscr{O}_{d2}}|+|P_{\mathscr{O}_{e2}}|+|P_{\varepsilon_2}|+|\mathscr{P}_{P_{S^*}}|. \tag{11}$$

Summing up (7), (8) and (13), then $\frac{5}{3}(l+2)(|\mathscr{I}_1|+|\mathscr{D}_1|+|\mathscr{I}_D|) - 4(d_2(S)-|P_0|)$ $\leq |P_{\varepsilon_D}|+|P_{\mathscr{I}_2}|+|P_{\mathscr{D}_1}|+|P_{\mathscr{I}_1}|+|P_{\mathscr{I}_D}|+|P_{\mathscr{D}_2}|+|P_{\mathscr{O}_{d2}}|+|P_{\mathscr{O}_{e2}}|+|P_{\varepsilon_2}|+|\mathscr{P}_{P_{S^*}}|$. Clearly, $\frac{5}{3}(l+2)(|\mathscr{I}_1|+|\mathscr{D}_1|+|\mathscr{I}_D|) - 4(d_2(S)-|P_0|) \leq N$.

Divide both sides by $\frac{5}{3}(l+2)$, thus

$$(|\mathscr{I}_1|+|\mathscr{D}_1|+|\mathscr{I}_D|) \leq \frac{N}{\frac{5}{3}(l+2)} + \frac{12}{5(l+2)}(d_2(S)-|P_0|). \tag{12}$$

By (4)and (12), we obtain the result as follows:

$$d_0(S) - d_2(S) - 2|P_{S^*}| \leq \frac{N}{\frac{5}{3}(l+2)} + \frac{12}{5(l+2)}(d_2(S)-|P_0|) - (d_2(S)-|P_0|).$$

thus,

$$d_0(S) - d_2(S) - 2|P_{S^*}| \leq \frac{N}{\frac{5}{3}(l+2)}.$$

From the above results, we can prove:

$$\frac{d_0(S)-d_2(S)-2|P_{S^*}|}{N} = \varepsilon(S) \le \frac{1}{\frac{5}{3}(l+2)}.$$

Now the proof is completed.

Theorem 2 For any solution S returned by $PMM(l)$, we have $\varepsilon(S) \le \frac{1}{\frac{5}{3}(l+2)}$.

4 Conclusion

We improved the analysis for the algorithm in [6] with preprocessing phase for a network of a general topology and proved $PMM(l) = OPT + \frac{1}{2}N(1+\varepsilon)$, where $\frac{1}{2l+3} \le \varepsilon \le \frac{1}{\frac{5}{3}(l+2)}$. The lower bound has been proved in [8]. For any given ε, we reduce the gap between the lower bound and the upper bound by proving a better upper bound. This implies that further improving the analysis of the time complexity of the algorithm with preprocessing.

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