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A Continuous-time Inventory Model with Procurement from Spot Market

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Abstract Not only the amount of product demand but also the prices of the product have a strong impact on a manufacturer's revenue. In this paper we consider a continuous-time inventory model where the spot price of the product stochastically fluctuates according to a Brownian motion. Should information of the spot price be available, the manufacturer wishes to buy the product from the spot market if profitable. The purpose of this paper is to find an optimal procurement policy so as to minimize the total expected discounted costs over an infinite planning horizon. We extend Sulem(1986) model into the one in which not only demand but also the market price of the product follows geometric Brownian motions. Then we obtain the optimal cost as the solution of Quasi-variational inequality, and show that there exists an optimal procurement policy as (s, S) policy. We shall clarify the dependence of such optimal (s, S) policy on the spot price at the procurement epoch. These values of (s, S) policy can be used and revised in the succeeding ordering cycles. Finally, some numerical examples are provided to investigate analytical properties of the expected cost function as well as of the optimal policy.

Keywords Inventory control; Stochastic model; Quasi-variational inequality

1 Introduction

Many manufactures that use of the spot market to procurement in supply chains are facing a fluctuation of market price. In this paper we consider a continuous-time inventory model in which the spot price of the product stochastically fluctuates according to a Brownian motion. The inventory level is being monitored on a continuous basis. Our objective is to determine the procurement policy as (s, S) policy to reduce the risk of spot price. When the inventory level drops to the reorder point, a pair of order quantity and reorder point for the next cycle is determined after observing the spot price.

Price uncertainty has been taken into account by several researchers in the context of inventory policy. Goel and Gutierrez [1] considered the value of incorporating information about spot and futures market prices in procurement decision making. On the other hand, there are many articles [2], [3], [4], [5] related to deal with (s, S) policy under the continuous time. Sulem [5] analyzed the optimal ordering policy for impulse control of an inventory system subject to a demand followed by a diffusion process. Furthermore, Benkherouf [4] further extend Sulems model to the case of general storage and shortage penalty cost function. We extend Sulem model into the one in which the market price of the product follows geometric Brownian motions, but demand is deterministic.

The reminder of this paper is organized as follows. In section 2 we formulate the model. Section 3 presents the optimal cost as the solution of Quasi-variational inequality, and show in section 4 that there exists an optimal procurement policy as (s, S) policy. Finally, some numerical examples are provided to investigate the analytical properties of the expected cost function as well as of the optimal policy.

2 Notation and Assumption

The analysis is based on the following assumptions:

(i) Time is continuous and inventory is continuously reviewed.
(ii) Demand is g units per unit time in one cycle. Unsatisfied demand is backlogged.
(iii) A critical-level (s,S,y) policy is in place, which means that the inventory level x, is drops to reorder point s, then the inventory level increases up to the order quantity S. And then the next s and S are determined, based on the observation of the spot price at time t, $y(t)$. Since s and S are changing at the beginning of each cycle, we suppose that $\bar{s}$ and $\overline{S}$ are reorder point and order quantities for last cycle, respectively. Therefore, $\overline{S}$ represents the initial inventory level at the beginning of the next cycle.
(iv) The set up cost is K and unit cost is equal to spot price $y(\cdot)$. The shortage cost p and holding cost q are given by the function:

$$f(x) = \begin{cases} -px & \text{for } x < 0, \\ qx & \text{for } x \geq 0. \end{cases} \tag{1}$$

(v) The spot price at time t, $y(t)$, follows a geometric Brownian motion, that is,

$$dy(t) = \mu y(t)dt + \sigma y(t)dw(t). \tag{2}$$

where $w(t)$ follows a Wiener process.

A procurement policy consists of a sequence $V = \{(\theta_i, \xi_i), i = 1, 2, \cdots\}$ of ordering times θ_i and order quantity ξ_i. Let $u(y,x)$ be the optimal total expected discounted cost over an infinite planning horizon when an initial inventory level is given by x and the spot price by y, are discounted at interest rate $\alpha > 0$. Then $u(y,x)$ can be written as

$$u(y,x) = \inf_V \left(E_y \left[\int_0^\infty f(x(t))e^{-\alpha t}dt + \sum_{i\geq 1}(K + y(\theta_i)\xi_i)e^{-\alpha\theta_i} \right] \Bigg| x(0) = x, y(0) = y \right) \tag{3}$$

where the inventory level $x(t)$ is given by

$$x(t) = x - gt + \sum_{i\geq 1} \xi_i \int_0^t \delta(s - \theta_i)ds. \tag{4}$$

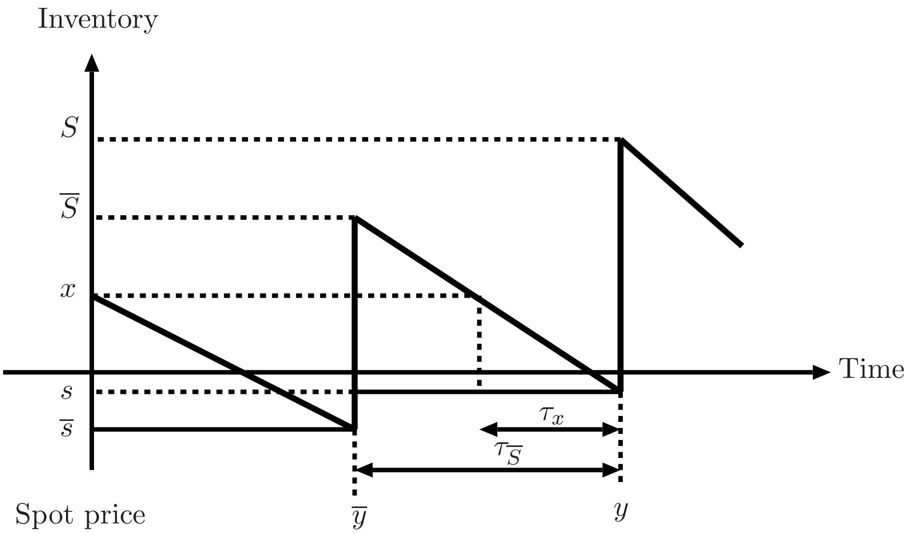

Figure 1: Inventory flow

with $x(0) = x$. In equation (4), $\delta(\cdot)$ denotes the Dirac function, that is,

$$\begin{aligned} \delta(z) &= 0 \ \text{ for } z \neq 0, \\ \int_{\infty}^{\infty} \delta(z)dz &= 1. \end{aligned}$$

We assume that u is continuous and almost everywhere differentiable in y. Note that $u(y,x)$ is continuous and everywhere differentiable in x. It is assumed that $u(y,x)$ is twice differentiable in y.

Our objective is to find the optimal procurement policy (s, S) at which the minimum value function u be attainedĄD

3 QVI Problem and Optimal Procurement Policy

In this section, we deal with equation (3) as a Quasi-Variational Inequality (QVI) problem (Bensoussan and Lions [6]).

First, if the procurement is not made at least during a small time interval $(t, t+\varepsilon)$, then we have following inequation;

$$u(y,x) \leq \int_{t}^{t+\varepsilon} f(x(s))e^{-\alpha(s-t)}ds + u(y(t+\varepsilon), x(t+\varepsilon))e^{-\alpha\varepsilon}. \tag{5}$$

We can expand the right hand side of equation (5) up to first order in ε and then we have

$$\varepsilon f(x) + u(y,x) - g\varepsilon \frac{\partial u}{\partial x} + \mu y \varepsilon \frac{\partial u}{\partial y} + \frac{1}{2}\varepsilon \sigma^2 y^2 \frac{\partial^2 u}{\partial y^2} - \alpha \varepsilon u(y,x) + o(\varepsilon^2). \tag{6}$$

Hence, making ε tend to 0, equation (5) deduce to

$$\alpha u + g\frac{\partial u}{\partial x} - \mu y\frac{\partial u}{\partial y} - \frac{1}{2}\sigma^2 y^2\frac{\partial^2 u}{\partial y^2} \leq f(x). \tag{7}$$

On the other hand, if the procurment is made at time t, the inventory level jumps from x to an $x+\xi$. We assume that the order quantity is delivered immediately, so the spot price is not change before as well as after procurement. Thus, we obtain

$$u(y,x) \leq K + \inf_{\xi\geq 0}(y(t)\xi + u(x+\xi, y(t))). \tag{8}$$

Therefore, the equation (3) is given by a solution of the QVI problem:

$$\begin{cases} Au \leq f \\ u \leq Mu \\ (Au-f)(u-Mu) = 0 \end{cases} \tag{9}$$

where

$$Au(y,x) := \alpha u + g\frac{\partial u}{\partial x} - \mu y\frac{\partial u}{\partial y} - \frac{1}{2}\sigma^2 y^2\frac{\partial^2 u}{\partial y^2}, \tag{10}$$

$$Mu(y,x) := K + \inf_{\xi\geq 0}(y(t)\xi + u(x+\xi, y(t))). \tag{11}$$

4 Solution of the QVI

In this section we solve the QVI quoted by Sulem [5] in part. We divide the inventory space into two regions; for no procurement,

$$G = \{x \in \mathscr{R} : u(y,x) < Mu(y,x)\} = \{x \in \mathscr{R} : x > s\} \tag{12}$$

then, we have

$$Au = f. \tag{13}$$

And its complement is given by

$$\overline{G} = \{x \in \mathscr{R} : u(y,x) = Mu(y,x)\} = \{x \in \mathscr{R} : x \leq s\} \tag{14}$$

and so, we have

$$u(y,x) = K + \inf_{\xi\geq 0}(y(t)\xi + u(y(t), x+\xi)) \tag{15}$$

$$= K + y(t)(S-x) + u(y(t), S). \tag{16}$$

since u is continuously differentiable in x, in inventory space $\overline{G}$, we can get the boundary conditions on u.

(i) Continuity of the derivative of u at the boundary point s:

$$\left.\frac{\partial u(y,x)}{\partial x}\right|_{x=s} = -y. \tag{17}$$

(ii) The infimum in equation (15) is attained at $\overline{S}$:

$$\left.\frac{\partial u(\overline{y},x)}{\partial x}\right|_{x=\overline{S}} = -\overline{y}, \tag{18}$$

where $\overline{y}$ is the spot price at the beginning of the cycle.

(iii) u is continuous at s:

$$u(S,y) = u(s,y) - K - y(S-s). \tag{19}$$

(iv) The growth condition of u:

$$\lim_{x\to\infty} \frac{u(x,y)}{f(x)} < +\infty. \tag{20}$$

To obtain the value function u, we solve the partial differential equation (13) with the initial and boundary conditions (17)-(20). First, we set

$$v(z,\tau) = u(y,x), \quad v(z,\tau) = w(z,\tau)e^{kz+l\tau}, \quad z = \log y, \quad \tau = \frac{x-s}{g}. \tag{21}$$

This results in the equation

$$\left(l - \mu k + \alpha + \frac{1}{2}k(1-k)\sigma^2\right) w + \frac{\partial w}{\partial \tau} + \left(\left(\frac{1}{2} - k\right)\sigma^2 - \mu\right)\frac{\partial w}{\partial z} - \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial z^2} = f(g\tau + s)e^{-(kz+l\tau)}. \tag{22}$$

where we choice

$$k = -\frac{1}{\sigma^2}\left(\mu - \frac{1}{2}\sigma^2\right), \quad l = -\frac{1}{2\sigma^2}\left(\mu - \frac{1}{2}\sigma^2\right)^2 - \alpha. \tag{23}$$

Then, we can get the non-homogeneous heat equation

$$\frac{\partial w}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial z^2} = f(g\tau+s)\exp\left\{\frac{1}{\sigma^2}\left(\mu - \frac{1}{2}\sigma^2\right)z + \left(\frac{1}{2\sigma^2}\left(\mu - \frac{1}{2}\sigma^2\right)^2 + \alpha\right)\tau\right\} \tag{24}$$

with

$$w(z,0) = e^{\frac{1}{\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)z}\{v(z,\tau_S) + K + e^z(S-s)\} \equiv m(z), \tag{25}$$

$$\tau_S = \frac{S-s}{g}. \tag{26}$$

The solution to the diffusion equation problem is given by

$$w(z,\tau) = w_1(z,\tau) + w_2(z,\tau), \tag{27}$$

where $w_1(z,\tau)$ and $w_2(z,\tau)$ are solutions of following problems:

$$\frac{\partial w_1}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 w_1}{\partial z^2}, \;\; w_1(z,0) = m(z), \tag{28}$$

$$\frac{\partial w_2}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 w_2}{\partial z^2} + h(z,\tau), \;\; w_2(z,0) = 0. \tag{29}$$

Here, we set the right hand side of equation (24) as $h(z,\tau)$. The solutions w_1, w_2 of each problem (28) and (29) are, respectively, given by

$$\begin{aligned} w_1(z,\tau) &= \frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{\infty} w_1(\xi,0)\exp\left\{-\frac{(z-\xi)^2}{2\sigma^2\tau}\right\}d\xi \\ &= \frac{1}{\sqrt{2\pi}}e^{n_-(z,\eta)}\int_{-\infty}^{\infty} v\left(\tau\left(\mu-\frac{\sigma^2}{2}\right)+\lambda\sigma\sqrt{\tau}+z,\tau_S\right)e^{-\frac{\lambda^2}{2}}d\lambda \\ &\quad +Ke^{n_-(z,\eta)}+(S-s)e^{n_+(z,\eta)}, \end{aligned} \tag{30}$$

and

$$\begin{aligned} w_2(z,\tau) &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_0^{\tau}\frac{1}{\sqrt{\tau-\delta}}\left(\int_{-\infty}^{\infty} h(\xi,\delta)\exp\left(-\frac{(z-\xi)^2}{2\sigma^2(\tau-\delta)}\right)d\xi\right)d\delta \\ &= e^{n_-(z,\tau)}\int_0^{\tau} f(g\delta+s)e^{\alpha\delta}d\delta, \end{aligned} \tag{31}$$

where

$$n_{\pm}(z,\tau) = \frac{1}{2\sigma^2}\left(\mu \pm \frac{\sigma^2}{2}\right)\left(\tau\left(\mu \pm \frac{\sigma^2}{2}\right)+2z\right). \tag{32}$$

Therefore, from equations (21), (27), (30) and (31), $u(y,x)$ can be rewritten as follows;

$$\begin{aligned} u(y,x) = e^{-\alpha\tau}\Bigg[&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(ye^{\tau_x\left(\mu-\frac{\sigma^2}{2}\right)+\lambda\sigma\sqrt{\tau_x}},S)e^{-\frac{\lambda^2}{2}}d\lambda \\ &+K+(S-s)ye^{\mu\tau_x}+\int_0^{\tau_x} f(g\delta+s)e^{\alpha\delta}d\delta\Bigg] \end{aligned} \tag{33}$$

where $\tau_x = (x-s)/g$. In the last term of equation (33), it can be rewritten as

$$\begin{aligned} D(x) &\equiv \int_0^{\tau_x} f(g\delta+s)e^{\alpha\delta}d\delta \\ &= \begin{cases} \frac{p}{\alpha}\left(s-\frac{g}{\alpha}\right)+\frac{q}{\alpha}\left(x-\frac{g}{\alpha}\right)e^{\alpha\tau_x}+\frac{g}{\alpha^2}(p+q)e^{-\frac{\alpha}{g}s} & \text{for } x \geq 0, \\ -\frac{p}{\alpha}\left\{\left(x-\frac{g}{\alpha}\right)e^{\alpha\tau_x}-s+\frac{g}{\alpha}\right\} & \text{for } x < 0. \end{cases} \end{aligned} \tag{34}$$

Therefore, we have

$$\begin{aligned} u(y,x) &= e^{-\alpha\tau}[E[u(ye^{\tau_x(\mu-\frac{\sigma^2}{2})+\sigma\sqrt{\tau_x}X},S)]+K+(S-s)ye^{\mu\tau_x}+D(x)] \\ &= e^{-\alpha\tau}[u(y_S,S)+K+(S-s)ye^{\mu\tau}+D(x)] \end{aligned} \tag{35}$$

where X is a standard normal random variable and y_S is the spot price for inventory level $x = S$.

Lemma 1. *The optimal cost function $u(y,x)$ is given by*

$$u(y,x) \quad = \quad \frac{g}{\alpha}\left(\frac{q}{g}\overline{S}+\overline{y}\right)e^{\alpha(\tau_{\overline{S}}-\tau_x)}+\left\{ye^{(\mu-\alpha)\tau_x}-\left(1-\frac{\mu}{\alpha}\right)\overline{y}e^{\mu\tau_{\overline{S}}-\alpha\tau_x}\right\}(S-s)+L(x) \tag{36}$$

where

$$L(x) \quad = \quad \begin{cases} \frac{q}{\alpha}\left\{\left(x-\frac{g}{\alpha}\right)-\left(\overline{S}-\frac{g}{\alpha}\right)e^{\alpha(\tau_{\overline{S}}-\tau_x)}\right\} & \text{for } x \geq 0, \\ -\frac{p}{\alpha}\left(x-\frac{g}{\alpha}\right)-\frac{g}{\alpha}\left(\overline{S}-\frac{g}{\alpha}\right)e^{\alpha(\tau_{\overline{S}}-\tau_x)}-\frac{g}{\alpha^2}(p+q)e^{-\frac{\alpha}{g}x} & \text{for } x < 0. \end{cases} \tag{37}$$

Proof. From equation (17), we have

$$u(\overline{y},\overline{S}) = \frac{g}{\alpha}\left(\frac{q}{g}\overline{S}+\overline{y}\right)e^{\alpha\tau_{\overline{S}}}-\left(1-\frac{\mu}{\alpha}\right)(S-s)\overline{y}e^{\mu\tau_{\overline{S}}}-K-D(\overline{S}). \tag{38}$$

Equations (38) and (35) lead to equation (36). □

Remark 2. Note that the equation (36) can be reduced to the deterministic-demand case of Sulem's model when we assume $\mu = 1$, $\sigma = 0$, $y = \overline{y}$, $S = \overline{S}$. In this case, the optimal cost $\tilde{u}(x)$ can be reduced to be

$$\tilde{u}(x) \quad = \quad \left\{\frac{(q+\alpha y)g}{\alpha^2}e^{\frac{\alpha\tilde{S}}{g}}-\frac{qg}{\alpha^2}\right\}e^{-\frac{\alpha x}{g}}+\frac{r}{\alpha}x+\frac{rg}{\alpha^2}\left(e^{-\frac{\alpha x}{g}}-1\right), \tag{39}$$

where

$$r \quad = \quad \begin{cases} -p & \text{for } x < 0, \\ q & \text{for } x \geq 0. \end{cases} \tag{40}$$

Lemma 2. *If an optimal order point s exists for any y, s is strictly negative.*

Proof. From equation (36), we have

$$\frac{\partial u}{\partial y} = e^{(\mu-\alpha)}(S-s), \quad \frac{\partial^2 u}{\partial y^2} = 0. \tag{41}$$

Therefore, $Au = y$ can be rewritten as

$$-\mu y e^{(\mu-\alpha)}(S-s)+g\frac{\partial u}{\partial x}+\alpha u = f(x). \tag{42}$$

Here, we assume that $s \geq 0$ and we can obtain $\partial u(y,S)/\partial x = -y$ from the same type of reasoning for equation (18). Moreover, $S > s$ and equation (17) yields that $u(y,S) > u(y,s)$. It is a contradiction with equation (19). Therefore, s is strictly negative. □

Theorem 1. *If $p > (\alpha - \mu)y$, then an optimal policy (s, S) is solution of the QVI problem, and the value of (s, S) is given by the solution of following simultaneous equation:*

$$0 = \frac{g}{\alpha}\left\{\frac{p}{g}s - y + \left(\frac{q}{g}\overline{S} + \overline{y}\right)e^{\alpha(\tau_{\overline{S}} - \tau_S)}\right\} + \left\{ye^{(\mu-\alpha)\tau_S} + \left(1 - \frac{\mu}{\alpha}\right)\left(y - \overline{y}e^{\mu\tau_{\overline{S}} - \alpha\tau_S}\right)\right\}(S - s) + K \quad (43)$$

$$\left\{\frac{g}{\alpha}(\frac{q}{g}\overline{S} + \overline{y})e^{\alpha\tau_{\overline{S}}} - (1 - \frac{\mu}{\alpha})(S - s)\overline{y}e^{\mu\tau_{\overline{S}}}\right\}(1 - e^{-\alpha\tau_S}) = D(\overline{S}) + ye^{(\mu-\alpha)\tau_S}(S - s) + K \quad (44)$$

Proof. Equation (43) and (44) are derived from equation (17) and (18), respectively. The proof is similar to the proof of Theorem 1 in Benkherouf [4]. □

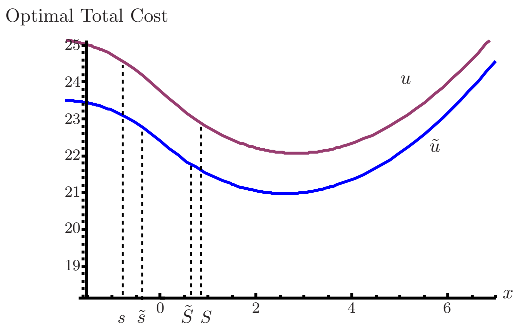

Figure 2: Optimal cost function for spot and sulem model

5 Numerical Examples

In this section we present the numerical example of optimal procurement policy and illustrate the value function. Since the equations (43) and (44) depend on the spot price y at the end of cycle, we assume that the price is equal to the expectation of spot price at the end of cycle in order to estimate the total cost at the beginning of cycle. Thus, we assign $y \approx \overline{y}\exp\{(\mu + \sigma^2/2)(\tau_{\overline{S}} - \tau_x)\}$ to equation (36) and recalculate equations (43) and (44) to implement the numerical computing. We assume that $g = 0.2$, $p = 0.12$, $q = 0.08$, $K = 0.14$, $\alpha = 0.01$, $\mu = 0.01$, $\sigma = 0.12$, $\overline{y} = 0.85$ and $\overline{S} = 0.586$. Then, we have $S = 0.7220$, $s = -0.823$, $\tilde{S} = 0.586$, $\tilde{s} = -0.478$. We plot the optimal cost function u with respect to x and $\overline{y}$ in Figure 2. Figure 3 shows the optimal cost function u and $\tilde{u}$ that the optimal cost for spot model is higher than the sulem's one result in the fluctuation of spot price. In Figure 4 an optimal procurement policies (s, S) illustrated for the drift of spot price from $\mu = 0$ to $\mu = 0.07$. The reorder point s is decreasing in μ for [0, 0.04] which means that it is better to backlog the demand than procure product immediately. For more large value of μ, the order quantity is decreasing.

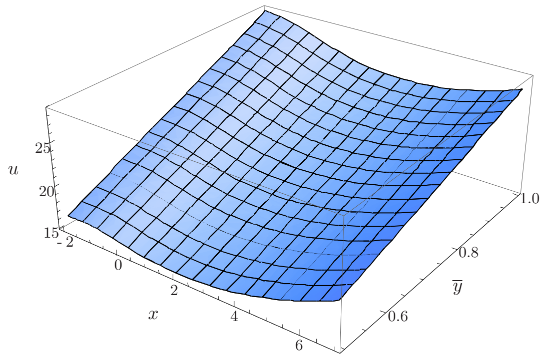

Figure 3: Optimal cost function with respect to x and $\overline{y}$

6 Concluding Remarks and Further Research

In this paper, we showed that an optimal policy for the inventory model that permits the manufactures to procure the products from spot market. We obtained the optimal cost function as the solution of Quasi-variational inequality, and showed that the exists the optimal procurement policy. For future research we extend the inventory model with procurement from spot market into the model that demand is also follows diffusion process and incorporate the supply contracts like option or futures into the model.

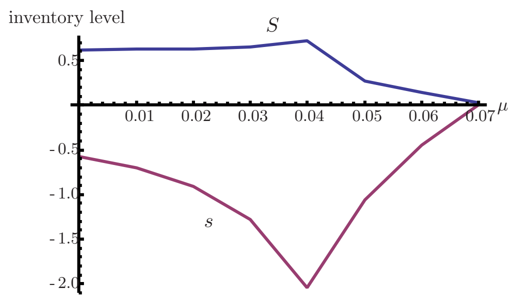

Figure 4: The optimal procurement policy with respect to μ

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