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Analysis of a Geom/G/1 Queue with General Limited Service and MAVs

Zhanyou Ma[1] Wuyi Yue[2] Naishuo Tian[1]

[1]College of Science, Yanshan University, Qinhuangdao 066004, China
[2]Department of Intelligence and Informatics, Konan University, Kobe 658-8501, Japan

Abstract In this paper, we present an analysis for a general limited service Geom/G/1 queue model with multiple adaptive vacations (MAVs). The Probability Generating Function (P.G.F.) of the queue length is obtained by using an embedded Markov chain with a regeneration cycle approach. The P.G.F. of the waiting time is also derived based on the independence between the arrival process and the waiting time. The probabilities for the system being in various states, such as "busy", "idle" and "vacation", are also derived.

Keywords Multiple adaptive vacations; general limited service; embedded Markov chain method; regeneration cycle approach; service cycle.

1 Introduction

Many Geom/G/1 queues with various vacation policies have been well investigated. Early researches focused on exhaustive service policy where the server takes a vacation only if the system becomes idle [1], [2]. However, multifarious non-exhaustive service policies also have important practical value in computer systems and communication networks [3], [4].

Several non-exhaustive service policies are introduced into the discussion of performance analysis of polling systems in [5], [6], where the server may take a vacation when there are customers waiting in the system. Stochastic decomposition results of vacation queues with general vacation policies are proved in [7], [8]. Geom/G/1 queues with a variety of vacation policies are systemically analyzed in [9]. Research results of multi-server vacation queues are studied in [10].

In the adaptation of different application background, some new vacation policies were introduced into queues. A class of Geom/G/1 queues with exhaustive service and multiple adaptive vacations were studied in [10]. A discrete time Geom/G/1 queue with multiple adaptive vacations was investigated in [11]. The multiple adaptive vacation policy is a synthetic policy which generalizes several simple vacation policies.

In this paper, we study a Geom/G/1 queue with non-exhaustive service and multiple adaptive vacations. Using an embedded Markov chain method and the regeneration cycle approach, we obtain the transformation formulae of the stationary queue length and the waiting time, and give stochastic decomposition structures of these stationary performance indices. General limited service Geom/G/1 queues with multiple vacations or single vacation in [9] are the special cases of our model.

2 System Model

Considering a classical Geom/G/1 queue, we introduce a general limited service and multiple adaptive vacation policy [10], [11]: a general limited service policy means that the number of customers which is served in every service period is not more than a determinate upper limit M, M is a positive integer.

$Q_b^{(n)}$ is the number of customers in the system at the beginning instant of the nth service period, then the number of customers who will be served in the next service period is given by

$$\Phi = \min\left\{Q_b^{(n)}, M\right\}.$$

The server will take H vacations with random length consecutively according to the assistant workload completed at that time. The number of the vacations is denoted by H. H is a positive integer random variable with the probability distribution h_j and the Probability Generating Function (P.G.F.) $H(z)$ as follows:

$$P(H = j) = h_j, \quad j \geq 1, \quad H(z) = \sum_{j=1}^{\infty} h_j z^j.$$

Vacation time lengths V_k $(k = 1, 2, ..., H)$ are independently identically distributed (i.i.d.) random variables. There are three cases in this system model as follows:

(1) If there are customers arriving during the kth vacation, $1 \leq k \leq H$, the vacation period will stop in advance at the completion instant of the kth vacation. The server will begin a new service period, and the server will take vacations again until Φ customers are served.
(2) If there are no customers arriving during H vacations, the server decides whether or not to enter an idle period based on the number of residual customers at the beginning instant of the service period after the Hth vacation finishes. If there are residual customers in the system, the server serves these residual customers immediately, then takes vacations.
(3) If there are no residual customers in the system after the Hth vacation finishes, the system enters an idle period. When customer arrives, the server emerges from an idle state to serve the customer, then it begins to take vacations. The system will continually repeat the above process.

The basic assumptions of the system model are given as follows:

(1) Suppose that customer arrivals can only occur at discrete time instant $t = n^-, n = 0, 1, ...$ The service starts and ends at discrete time instant $t = n^+$ only, $n = 1, 2, ...$ The model is called a late arrival system. The inter-arrival time T is supposed to be an independently identically distributed (i.i.d.) discrete random variable following a geometric distribution with parameter λ $(0 < \lambda < 1)$. We can write the probability distribution of T as follows:

$$P(T = j) = \lambda \bar{\lambda}^{j-1}, \quad j = 1, 2, ...$$

where $\bar{\lambda} = 1 - \lambda$. We denote by C_n the number of customers arriving during the interval $[0,n]$, then C_n follows a Binomial distribution as follows:

$$P(C_n = j) = \binom{n}{j} \lambda^j \bar{\lambda}^{n-j}, \quad j = 0,1,...,n.$$

(2) The service time S of a customer is supposed to be an i.i.d. discrete random variable with a general distribution s_j. The P.G.F. $S(z)$, the mean $E[S]$ and the second factorial moment $E[S(S-1)]$ of S are given as follows:

$$P(S_i = j) = s_j, \quad j \geq 1, \quad S(z) = \sum_{j=1}^{\infty} s_j z^j.$$

$$E[S] = \sum_{i=0}^{\infty} i s_i, \quad E[S(S-1)] = \left. \frac{d^2 S(z)}{dz^2} \right|_{z=1}.$$

Let μ be the reciprocal value of the mean $E[S]$, then we have $1/\mu = E[S]$.

(3) The time length V of a vacation is a nonnegative i.i.d. discrete random variable with general probability distribution v_j and the P.G.F. $V(z)$ as

$$P(V = j) = v_j, \quad j \geq 1, \quad V(z) = \sum_{j=1}^{\infty} v_j z^j$$

where $E[V]$ and $E[V(V-1)]$ exist. $\rho = \lambda/\mu$ is traffic intensity of the system.

Suppose that there is a single server in this system, and its capability is infinite. The inter-arrival time, the service time and the time length of a vacation are mutually independent. The service order is First Come First Served (FCFS). The model is denoted by Geom/G/1 (GL, MAVs), where GL and MAVs represent the General Limited service and the Multiple Adaptive Vacations, respectively.

3 Performance Analysis

3.1 Preliminaries

According to non-exhaustive service policy, the transition probability matrix of the queue length $\{L_n, n \geq 1\}$ at the departure instant of a customer is different from the transition probability matrix of a classical Geom/G/1 queue at not only boundary states but also all states, thus we can not simply apply the results of a Geom/G/1 boundary state variation model to study Geom/G/1 with the non-exhaustive service policy. The regeneration cycle approach is the most effective to apply to the stationary queue length of a system with a non-exhaustive service policy.

Let $L_v(t)$ represent the queue length process of a Geom/G/1 queue with a non-exhaustive service policy. The beginning instants of the service cycle are chosen as regeneration points when the number of customers is zero. The process $L_v(t)$ can be assumed to restart at these instants. If the system is positive recurrent, $L_v(t)$ will transit the zero state an infinite amount of times, therefore the system has infinite regeneration points. The interval between two adjacent regeneration points is defined as a regeneration cycle. A regeneration cycle may include several service cycles, and the length of a regeneration cycle is an i.i.d. random variable.

Lemma 1. (see [9, 10]) If a stationary distribution exists, its P.G.F. can be given as

$$L_v(z) = \frac{E\left[\sum_{n=1}^{\Phi} z^{L_n}\right]}{E[\Phi]} \tag{1}$$

where L_n denotes the number of customers in the system at the nth departure instant.

3.2 Number of Customers at the Beginning Instant

Transition probabilities of Markov chain $\{L_n, n \geq 1\}$ are given as follows:

$$P_{jk} = \begin{cases} \sum_{r=k-j+M}^{\infty} \binom{r}{k-j+M} \lambda^{k-j+M} \bar{\lambda}^{r-k+j-M} P\left(B^{(M)} + V = r\right), \\ \hfill k \geq j-M > 0 \\ (1-H(V(\bar{\lambda}))) \sum_{r=k}^{\infty} \binom{r}{k} \lambda^k \bar{\lambda}^{r-k} P\left(B^{(j)} + V = r\right), \quad j \leq M, k \neq 1 \\ (1-H(V(\bar{\lambda}))) \sum_{r=k}^{\infty} \binom{r}{k} \lambda^k \bar{\lambda}^{r-k} P\left(B^{(j)} + V = r\right) + H(V(\bar{\lambda})), \\ \hfill j \leq M, k = 1 \\ 0, \hfill j > M, k < j-M \end{cases} \tag{2}$$

where $B^{(j)} + V$ is the sum of j service times and V, $0 \leq j \leq M$. Define that

$$q_k = \lim_{n \to \infty} P\left(Q_b^{(n)} = k\right), \quad k \geq 0.$$

$\{q_k, k \geq 0\}$ is the distribution of Q_b customers in the system at the beginning instant of the service period. From the equilibrium equation of the Markov chain, we have

$$q_0 = V(\bar{\lambda})(1-H(V(\bar{\lambda}))) \sum_{j=0}^{M} q_j (S(\bar{\lambda}))^j, \tag{3}$$

$$\begin{aligned} q_1 = & \sum_{j=0}^{M} q_j \left((1-H(V(\bar{\lambda}))) \sum_{r=j}^{\infty} \binom{r}{1} \lambda \bar{\lambda}^{r-1} P\left(B^{(j)} + V = r\right) + H(V(\bar{\lambda})) \right) \\ & + q_{M+1} \sum_{r=j}^{\infty} \binom{r}{0} \lambda^0 \bar{\lambda}^r P\left(B^{(M)} + V = r\right), \end{aligned} \tag{4}$$

$$\begin{aligned} q_k = & \sum_{j=0}^{M} q_j (1-H(V(\bar{\lambda}))) \sum_{r=j}^{\infty} \binom{r}{k} \lambda^k \bar{\lambda}^{r-k} P\left(B^{(j)} + V = r\right) \\ & + \sum_{j=M+1}^{k+M} \sum_{r=M}^{\infty} \binom{r}{k-j+M} \lambda^{k-j+M} \bar{\lambda}^{r-k+j-M} P\left(B^{(M)} + V = r\right), \ k \geq 2. \end{aligned} \tag{5}$$

Define the partial probability generating function as follows:

$$Q_M(z) = \sum_{k=0}^{M} q_k z^k.$$

Multiplying both sides of Eqs. (3)-(5) by z^0, z, z^k, respectively, and taking the summation with respect to k, we obtain the P.G.F. of $\{q_k, k \geq 0\}$ as follows:

$$\begin{aligned}Q_b(z) = & \frac{1}{z^M - (S(1-\lambda(1-z)))^M V(1-\lambda(1-z))} \\ & \times \Big((1-H(V(\bar{\lambda})))z^M Q_M(S(1-\lambda(1-z)))V(1-\lambda(1-z)) \\ & -(S(1-\lambda(1-z)))^M Q_M(z)V(1-\lambda(1-z)) + H(V(\bar{\lambda}))Q_M(1)z^{M+1}\Big). \qquad (6)\end{aligned}$$

To determine $Q_b(z)$, we should obtain the coefficients $q_0, q_1, ..., q_M$ by using the Rouche theorem [12] and the Lagrange theorem [13]. In the denominator of Eq. (6), we define that

$$f(z) = z^M, \quad g(z) = -V(1-\lambda(1-z))(S(1-\lambda(1-z)))^M.$$

For the probability distribution $\{c_k, k \geq 0\}$ of any non-negative integer random variable X and a sufficiently small $\varepsilon > 0$, in $|z| = 1+\varepsilon$, we have

$$|C(z)| = \left| \sum_{k=0}^{\infty} c_k z^k \right| \leq \sum_{k=0}^{\infty} c_k (1+\varepsilon)^k = \sum_{k=0}^{\infty} c_k (1+k\varepsilon) + o(\varepsilon) = 1 + \varepsilon E[X] + o(\varepsilon).$$

Applying the above inequality to $g(z)$, we obtain

$$|g(z)| \leq 1 + (M\rho + \lambda E[V])\varepsilon + o(\varepsilon), \quad |f(z)| = (1+\varepsilon)^M = 1 + M\varepsilon + o(\varepsilon).$$

If $\rho + \lambda E[V]M^{-1} < 1$, then $|f(z)| > |g(z)|$ in $|z| = 1+\varepsilon$. According to the Rouche theorem, $f(z)$ and $f(z)+g(z)$ have the same number of roots in $|z| = 1+\varepsilon$. Therefore, the denominator of Eq. (6) has M roots in $|z| = 1+\varepsilon$, where one root is $z = 1$, the other $M-1$ roots are given by applying the Lagrange theorem as

$$z_m = \sum_{n=1}^{\infty} \frac{e^{\frac{2\pi mn}{M} i}}{n!} \frac{d^{n-1}}{dz^{n-1}} \left(V(1-\lambda(1-z))(S(1-\lambda(1-z)))^M \right)^{\frac{n}{M}} \Bigg|_{z=0} \qquad (7)$$

where $m = 1, 2, ..., M-1$. Because $Q_b(z)$ is analytic in $|z| < 1$, the numerator of Eq. (6) has the same roots. q_k satisfies the set of equations comprised of the following $M-1$ linear equations:

$$\begin{aligned}\sum_{k=0}^{M} q_k \Big(& (1-H(V(\bar{\lambda})))z_m^M (S(1-\lambda(1-z_m)))^k V(1-\lambda(1-z_m)) + H(V(\bar{\lambda}))z_m^{M+1} \\ & -(S(1-\lambda(1-z_m)))^M z_m^k V(1-\lambda(1-z_m)) \Big) = 0, \quad m = 1, ..., M-1. \qquad (8)\end{aligned}$$

Based on the normalization condition $Q_b(1) = 1$ and by applying the L'Hospital rule in Eq. (6), we have

$$1 = \frac{(M(1-\rho) + H(V(\bar{\lambda}))(1-\lambda E[V]))Q_M(1) - ((1-\rho) + \rho H(V(\bar{\lambda})))Q'_M(1)}{M(1-\rho) - \lambda E[V]}. \qquad (9)$$

From Eq. (9), we obtain the relation between $Q'_M(1)$ and $Q_M(1)$ as follows:

$$Q'_M(1) = \frac{\lambda E[V] - M(1-\rho)}{(1-\rho)+\rho H(V(\bar{\lambda}))} + \frac{M(1-\rho)+H(V(\bar{\lambda}))(1-\lambda E[V])}{(1-\rho)+\rho H(V(\bar{\lambda}))} Q_M(1). \qquad (10)$$

According to Eq. (10), we obtain the Mth equation about q_k as follows:

$$\sum_{k=0}^{M} q_k \left(M(1-\rho)+H(V(\bar{\lambda}))(1-\lambda E[V]) + k((1-\rho)+\rho H(V(\bar{\lambda})))\right) = M(1-\rho) - \lambda E[V]. \qquad (11)$$

From Eqs. (3), (8) and (11), we can obtain set of equations comprised by $M+1$ linear equations, thus we can resolve $q_k, k = 0,1,...,M$ and $Q_M(z)$. Taking derivatives for both sides of Eq. (6) with respect to z and by applying the L'Hospital rule, the mean number of customers at the beginning instant of the service cycle is given by

$$\begin{aligned} E[Q_b] = {} & \frac{1}{2(M(1-\rho)-\lambda E[V])} \Big\{ \lambda^2 M E[S(S-1)] \\ & - \Big(M(M-1)(1-\rho^2) + 2\lambda\rho M E[V] \\ & + \lambda^2 E[V(V-1)] \Big) + Q''_M(1)((1-H(V(\bar{\lambda})))\rho^2 - 1) \\ & + Q'_M(1) \Big((1-H(V(\bar{\lambda}))) \Big(\lambda^2 E[S(S-1)] + 2\lambda\rho E[V] \Big) \\ & - 2\rho M H(V(\bar{\lambda}))M - 2\lambda E[V] \Big) + Q_M(1) \Big((1-H(V(\bar{\lambda}))) \\ & \times \Big(M(M-1) + 2\lambda M E[V] + \lambda^2 E[V(V-1)] \Big) \\ & - M(M-1)\rho^2 - \lambda^2 M E[S(S-1)] - 2\lambda\rho M E[V] \\ & - \lambda^2 E[V(V-1)] + M(M+1)H(V(\bar{\lambda})) \Big) \Big\}. \end{aligned} \qquad (12)$$

Combining $\Phi = \min\{Q_b, M\}$ and Eq. (10), we obtain that

$$E[\Phi] = \frac{\lambda E[V] + \rho M H(V(\bar{\lambda})) + H(V(\bar{\lambda}))(1-\rho M - \lambda E[V]) Q_M(1)}{1-\rho(1-H(V(\bar{\lambda})))}. \qquad (13)$$

The equilibrium condition of the system requires that the mean number of customers arriving in a service cycle is less than M. Therefore, the equilibrium condition of the system is given by

$$M - \rho E[\Phi] - \lambda E[V] - H(V(\bar{\lambda}))(1-\lambda E[V]) > 0. \qquad (14)$$

Based on the regeneration cycle approach and the expression of $Q_b(z)$, we obtain the stochastic decomposition structure of stationary performance measures for a general limited service Geom/G/1 queue with multiple adaptive vacations.

3.3 Stationary Queue Length and Waiting Time

Theorem 1. If $M - \rho E[\Phi] - \lambda E[V] - H(V(\bar{\lambda}))(1 - \lambda E[V]) > 0$ and $\rho + \lambda E[V]M^{-1} < 1$, the stationary queue length L_v in Geom/G/1 (GL, MAVs) queue can be decomposed into three independent random variables:

$$L_v = L + L_d + L_r$$

where L is the stationary queue length in a classical Geom/G/1 queue [9], [10].

The additional queue length L_d is the additional queue length of a Geom/G/1 queue with multiple adaptive vacations. The additional queue length L_r is the additional queue length resulting from the general limited service policy. P.G.Fs. $L_d(z)$ and $L_r(z)$ of the additional queue lengths L_d and L_r are given by

$$L_d(z) = \frac{1 - zH(V(\bar{\lambda})) - \dfrac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}(V(1-\lambda(1-z)) - V(\bar{\lambda}))}{\left(H(V(\bar{\lambda})) + \dfrac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}\lambda E[V]\right)(1-z)}, \tag{15}$$

$$\begin{aligned} L_r(z) = & \frac{\beta}{z^M - (S(1-\lambda(1-z)))^M V(1-\lambda(1-z))} \\ & \times \frac{1}{1 - V(1-\lambda(1-z)) + H(V(\bar{\lambda}))\Big(V(1-\lambda(1-z)) - (1-V(\bar{\lambda}))z - V(\bar{\lambda})\Big)} \\ & \times \Big(Q_M(S(1-\lambda(1-z)))\left(z^M(1-V(1-\lambda(1-z)))\right. \\ & + H(V(\bar{\lambda}))V(1-\lambda(1-z))(z^M - (S(1-\lambda(1-z)))^M)\big) \\ & - Q_M(z)(S(1-\lambda(1-z)))^M(1-V(1-\lambda(1-z))) \\ & + H(V(\bar{\lambda}))Q_M(1)z\left((S(1-\lambda(1-z)))^M - z^M\right)\Big). \end{aligned} \tag{16}$$

Proof. Because L_n is the number of customers at the departure instant of the nth customer in a service cycle, and A_k represents the number of customer arriving in the kth service period, then we have

$$L_n = Q_b - n + \sum_{k=1}^{n} A_k, \quad n = 1, 2, ..., \Phi.$$

Substituting Eqs. (13) and $E\left[\sum_{n=1}^{\Phi} z^{L_n}\right]$ into Eq. (1), we obtain

$$L_v(z) = \frac{E\left[\sum_{n=1}^{\Phi} z^{L_n}\right]}{E[\Phi]} = L(z)L_d(z)L_r(z) \tag{17}$$

where

$$\beta = \frac{\left(1-\rho(1-H(V(\bar{\lambda})))\right)\left[\lambda E[V] + H(V(\bar{\lambda}))(1-V(\bar{\lambda}) - \lambda E[V])\right]}{(1-\rho)\left[\lambda E[V] + \rho M H(V(\bar{\lambda})) + H(V(\bar{\lambda}))Q_M(1)(1-\rho M - \lambda E[V])\right]}.$$

Taking a derivative of $L_v(z)$ with respect to z, then applying the L'Hospital rule repeatedly and letting $z = 1$, we obtain the mean additional queue length $E[L_v]$.

Based on the stochastic decomposition result of the queue length L_v and a classical relation, we can prove the stochastic decomposition result of the waiting time. The classical relation is that the number of customers in the system at the departure instant of a customer is equal to the number of customers arriving in the sojourn time of the customer.

Theorem 2. If $M-\rho E[\Phi]-\lambda E[V]-H(V(\bar{\lambda}))(1-\lambda E[V])>0$ and $\rho+\lambda E[V]M^{-1}<1$, the stationary waiting time W_v in Geom/G/1 (GL, MAVs) queue can be decomposed into three independent random variables:

$$W_v = W + W_d + W_r$$

where W is the stationary waiting time in a classical Geom/G/1 queue [9], [10].

The additional delay W_d is the additional delay of a Geom/G/1 queue with multiple adaptive vacations, the additional delay W_r is the additional delay resulting from the general limited service policy. P.G.Fs. $W_d(z)$ and $W_r(z)$ of additional delays W_d and W_r are given by

$$W_d(z)=\frac{\lambda-H(V(\bar{\lambda}))(z-\bar{\lambda})-\lambda\dfrac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}(V(z)-V(\bar{\lambda}))}{\left(H(V(\bar{\lambda}))+\dfrac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}\lambda E[V]\right)(1-z)}, \tag{18}$$

$$\begin{aligned}W_r(z)=&\frac{\beta}{(z-\bar{\lambda})^M-(\lambda S(z))^M V(z)}\\&\times\frac{1}{\lambda+H(V(\bar{\lambda}))(\bar{\lambda}-V(\bar{\lambda}))-\lambda V(z)(1-H(V(\bar{\lambda})))+H(V(\bar{\lambda}))(1-V(\bar{\lambda}))z}\\&\times\Big(zQ_M(S(z))\left((z-\bar{\lambda})^M(1-V(z))+H(V(\bar{\lambda}))V(z)((z-\bar{\lambda})^M\right.\\&\left.-(\lambda S(z))^M)\right)-\lambda^{M+1}Q_M\left(\frac{z-\bar{\lambda}}{\lambda}\right)(S(z))^M(1-V(z))\\&+H(V(\bar{\lambda}))Q_M(1)(z-\bar{\lambda})((\lambda S(z))^M-(z-\bar{\lambda})^M)\Big).\end{aligned} \tag{19}$$

Proof. Because the waiting time is independent of the input process after the arrival instant, the number of residual customers in the system after the departure instant is equal to the sum of the number of customers arriving in the waiting time W_v and the service time S for a Geom/G/1 (GL, MAVs) model. Because of the independent increment property of the input process which follows a binomial distribution, the number of customers arriving in the waiting time and the service time are mutually independent. We then have

$$L_v(z) = W_v(1-\lambda(1-z))S(1-\lambda(1-z)). \tag{20}$$

Substituting $L_v(z)$ in Theorem 1 into Eq. (20), and letting $z' = 1-\lambda(1-z)$, then displacing z' with z, we can obtain the P.G.Fs. $W_d(z)$ and $W_r(z)$ of additional delays W_d and W_r given by Eqs. (18) and (19), respectively.

Taking a derivative of Eqs. (18) and (19) with respect to z, and using the L'Hospital rule, we can obtain the mean waiting time $E[W_v]$.

3.4 Analysis of the Service Cycle

According to the definition of the number J of consecutive vacations [10] and [11], the P.G.F. $J(z)$ of J can be obtained as follows:

$$P(J \geq 1) = 1, \quad P(J \geq j) = (V(\bar{\lambda}))^{j-1} \sum_{k=j}^{\infty} h_k, \quad j \geq 2,$$

$$J(z) = 1 - \frac{1-z}{1-V(\bar{\lambda})z}(1 - H(V(\bar{\lambda})z)). \tag{21}$$

The P.G.F. $V_G(z)$ and the mean vacation time $E[V_G]$ of V_G which is the whole time length for every two consecutive vacations are given as follows:

$$V_G(z) = 1 - \frac{1-V(z)}{1-V(\bar{\lambda})V(z)}(1 - H(V(\bar{\lambda})V(z))), \quad E[V_G] = \frac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}E[V]. \tag{22}$$

In a general limited service Geom/G/1 queue with multiple adaptive vacations, the server may stay in an idle period. The idle period is equal to zero at the completion instant of J vacations when one of the following three cases holds: (i) there are customers arriving in the service period; (ii) there are no customers arriving in the service period but there are customers arriving during the vacation period; (iii) there are no customers arriving in the service period and the vacation period, but there are residual customers that have resulted from a general limited policy.

The idle period is equal to an inter-arrival time if there are no customer arriving at the completion instant of J vacations and there are no residual customers resulting from a general limited policy. Let I_v be the time length of the idle period, the mean $E[I_v]$ is given as follows:

$$E[I_v] = \frac{(1-\rho)\beta H(V(\bar{\lambda}))\left(H(V(\bar{\lambda}))V(\bar{\lambda})Q_M(S(\bar{\lambda})) + Q_M(0)(1-V(\bar{\lambda}))\right)}{\lambda V(\bar{\lambda})\left(H(V(\bar{\lambda}))(1-V(\bar{\lambda})) + \lambda E[V](1-H(V(\bar{\lambda})))\right)}. \tag{23}$$

The mean busy period is given as follows:

$$E[S_\lambda] = \frac{\lambda E[V] + \rho M H(V(\bar{\lambda})) + H(V(\bar{\lambda}))Q_M(1)(1 - M\rho - \lambda E[V])}{\mu(1-\rho(1-H(V(\bar{\lambda}))))}. \tag{24}$$

Let C be the interval between the beginning instant of two service periods, called a "service cycl". Then the mean service cycle $E[C]$ is given as follows:

$$E[C] = E[V_G] + E[I_v] + E[S_\lambda]. \tag{25}$$

Let p_B, p_V and p_I be the probabilities of the server is at the busy, vacation and idle states, respectively. We can give that

$$p_B = \frac{\lambda E[V] + \rho M H(V(\bar{\lambda})) + H(V(\bar{\lambda}))Q_M(1)(1 - M\rho - \lambda E[V])}{\mu(1-\rho(1-H(V(\bar{\lambda}))))E[C]},$$

$$p_V = \frac{E[V](1-H(V(\bar{\lambda})))}{E[C](1-V(\bar{\lambda}))},$$

$$p_I = \frac{(1-\rho)\beta H(V(\bar{\lambda}))\left(H(V(\bar{\lambda}))V(\bar{\lambda})Q_M(S(\bar{\lambda})) + Q_M(0)(1-V(\bar{\lambda}))\right)}{\lambda V(\bar{\lambda})E[C]\left[H(V(\bar{\lambda}))(1-V(\bar{\lambda})) + \lambda E[V](1-H(V(\bar{\lambda})))\right]}. \tag{26}$$

4 Conclusions

In this paper, we have presented an analysis of a general limited service Geom/G/1 queue with multiple adaptive vacations in details. We derived the P.G.Fs. of the stationary queue length and the waiting time by using an embedded Markov chain method and a regeneration cycle approach. Furthermore, we obtained the probabilities of the server being at the various states of busy, vacation and idle, respectively. Finally, some special cases were given to verify the results.

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