The 7th International Symposium on Operations Research and Its Applications (ISORA'08)
Lijiang, China, October 31–Novemver 3, 2008

An Arbitrary Starting Simplicial Algorithm for Constructively Proving Tarski's Fixed Point Theorem on an n-dimensional Box*

Chuangyin Dang[1,†]

[1]Dept. of Manufacturing Engineering & Engineering Management,
City University of Hong Kong, 83 Tat Chee Avenue Kowloon, Hong Kong

Abstract The well-known Tarski's fixed point theorem asserts that an increasing mapping from an n-dimensional box to itself has a fixed point. In this paper, a constructive proof of this theorem is obtained from an application of the (n+1)-ray arbitrary starting simplicial algorithm. The algorithm assigns an integer label to each point of the box and employs a triangulation to subdivide the box into simplices. For any given mesh size of the triangulation, starting from an arbitrary interior point of the box, the algorithm generates within a finite number of iterations a complete n-dimensional simplex, any point of which yields an approximate fixed point. If the accuracy is not good enough, the mesh size of the triangulation is refined and the algorithm is restarted. When the mesh size goes to zero sequentially, one will obtain a sequence of approximate fixed points satisfying that every limit point of the sequence is a fixed point.

Keywords Increasing Mapping; Fixed Point; Tarski's Fixed Point Theorem; Constructive Proof; Integer Labeling; Triangulation; Simplicial Algorithm.

1 Introduction

Let $N = \{1, 2, \cdots, n\}$. For any x and y of R^n, we say $x \leq y$ if $x_i \leq y_i$ for any $i \in N$. Let $H = \{x \in R^n \mid a \leq x \leq b\}$, where $a = (a_1, a_2, \cdots, a_n)^\top$ and $b = (b_1, b_2, \cdots, b_n)^\top$ are finite and $a_i < b_i$ for any $i \in N$. A mapping $f : H \to H$ is increasing if, for any x and y of H, $x \leq y$ implies $f(x) \leq f(y)$.

Theorem 1 (Tarski's Fixed Point Theorem).
If $f : H \to H$ is an increasing mapping, there exists $x^* \in H$ such that $f(x^*) = x^*$.

Simplicial algorithms for computing fixed points of a continuous or upper semi-continuous mapping were originated in Scarf (1967) and substantially developed in the literature (e.g., Allgower and Georg, 2000; Dang, 1991, 1995; Eaves, 1972, 1984; Eaves and Saigal, 1972; Forster, 1995; Garcia and Zangwill, 1981; Kojima and Yamamoto, 1984; Kuhn, 1968; van der Laan and Talman, 1979, 1981; Lüthi, 1975; Merrill, 1972; Scarf, 1973; Todd, 1976; Wright, 1981; Yamamoto, 1983). In this paper, simplicial algorithms have been extended to approximating a Traski's fixed point. A constructive proof

*This work was partially supported by CERG: CityU 101005 of the Government of Hong Kong SAR
†E-Mail: mecdang@cityu.edu.hk.

of Theorem 1 is obtained from an application of the (n+1)-ray arbitrary starting simplicial algorithm. The algorithm assigns an integer label to each point of the box and employs a triangulation to subdivide the box into simplices. For any given mesh size of the triangulation, starting from an arbitrary interior point of the box, the algorithm generates within a finite number of iterations a complete n-dimensional simplex, any point of which yields an approximate fixed point. If the accuracy is not good enough, the mesh size of the triangulation is refined and the algorithm is restarted. When the mesh size goes to zero sequentially, one will obtain a sequence of approximate fixed points satisfying that every limit point of the sequence is a fixed point.

The paper is organized as follows. In Section 2, an arbitrary starting simplicial algorithm is proposed to approximate a Tarski's fixed point in an n-dimensional box. In Section 3, a constructive proof of Tarski's theorem is obtained from an application of the algorithm in Section 2.

2 An Arbitrary Starting Simplicial Algorithm

In this section, an arbitrary starting simplicial algorithm will be developed to compute an approximate fixed point of f. It is derived from an application of the $(n+1)$-ray simplicial algorithm in van der Laan and Talman (1979).

Let x^0 be an arbitrary interior point of H, which will be the starting point of the algorithm. Let u^i be the ith unit vector of R^n and $h^i = -u^i$ for $i = 1, 2, \cdots, n$. Let $e = (1, 1, \cdots, 1)^\top \in R^n$ and $h^{n+1} = e$. Let $N_0 = \{1, 2, \cdots, n+1\}$. For any subset K of N_0 with $K \neq N_0$, let

$$G(x^0, K) = \{x^0 + \sum_{k\in K} \lambda_k h^k \mid 0 \leq \lambda_k,\ k \in K\}.$$

Then, $\cup_{K\subset N_0} G(x^0, K) = R^n$.

The K_1-triangulation will be chosen as the underlying triangulation of the algorithm, which is as follows. A simplex of the K_1-triangulation of R^n is the convex hull of $n+1$ integer vectors, $y^0, y^1, \ldots, y^n$, given by $y^0 = y$ and $y^k = y^{k-1} + u^{\pi(k)}$, $k = 1, 2, \ldots, n$, where y is an integer point of R^n and $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ a permutation of elements of $N = \{1, 2, \ldots, n\}$. Let K_1 be the set of all such simplices. Since a simplex of the K_1-triangulation is uniquely determined by y and π, we use $K_1(y, \pi)$ to denote it.

We say that two simplices of K_1 are adjacent if they have a common facet. We show how to generate all the adjacent simplices of a simplex of the K_1-triangulation of R^n in the following. For a given simplex $\sigma = K_1(y, \pi)$ with vertices y^0, y^1, …, y^n, its adjacent simplex opposite to a vertex, say y^i, is given by $K_1(\bar{y}, \bar{\pi})$, where $\bar{y}$ and $\bar{\pi}$ are generated in the following table.

Pivot Rules of the K_1-Triangulation

i	$\bar{y}$	$\bar{\pi}$
0	$y + u^{\pi(1)}$	$(\pi(2), \ldots, \pi(n), \pi(1))$
$1 < i < n$	y	$(\pi(1), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n))$
n	$y - u^{\pi(n)}$	$(\pi(n), \pi(1), \ldots, \pi(n-1))$

Let $\mathscr{K}_1$ be the set of faces of simplices of K_1. A q-dimensional simplex of $\mathscr{K}_1$ with vertices y^0, y^1, …, y^q is denoted by $< y^0, y^1, \ldots, y^q >$. The restriction of $\mathscr{K}_1$ on $G(x^0, K)$

for any subset $K \subset N_0$ is given by

$$\mathscr{K}_1|G(x^0,K) = \{\sigma \in \mathscr{K}_1 \mid \sigma \subset G(x^0,K) \text{ and } \dim(\sigma) = |K|\},$$

where $|\cdot|$ denotes the cardinality of a set and $\dim(\cdot)$ the dimension of a set. Obviously, $\mathscr{K}_1|G(x^0,K)$ is a triangulation of $G(x^0,K)$.

For $\sigma \in \mathscr{K}_1$, let $\text{grid}(\sigma) = \max\{\|x-y\|_\infty \mid x \in \sigma \text{ and } y \in \sigma\}$. We define $\text{mesh}(K_1) = \max_{\sigma\in\mathscr{K}_1} \text{grid}(\sigma)$. Clearly, $\text{grid}(\sigma) = 1$ for any $\sigma \in \mathscr{K}_1$ and $\text{mesh}(K_1) = 1$.

Let δ be any given positive number and

$$K_1(x^0,\delta) = \{x^0 + \delta\sigma \mid \sigma \in K_1\}.$$

Then, $K_1(x^0,\delta)$ is a triangulation of R^n with $\text{mesh}(K_1(x^0,\delta)) = \delta$.

For any $x \in R^n \backslash H$, let $f(x) = f(y)$ with

$$y = \text{argmin}_{z\in H}\|x-z\|_2.$$

Then, f becomes a mapping from R^n to H.

Lemma 1.
f is an increasing mapping on R^n.

Proof. Let x^1 and x^2 be any two points of R^n with $x^1 \le x^2$. Then, $y^1 = \text{argmin}_{z\in H}\|x^1-z\|_2$ is given by

$$y_i^1 = \begin{cases} x_i^1 & \text{if } a_i \le x_i^1 \le b_i, \\ a_i & \text{if } x_i^1 < a_i, \\ b_i & \text{if } x_i^1 > b_i, \end{cases}$$

for $i = 1,2,\cdots,n$, and $y^2 = \text{argmin}_{z\in H}\|x^2-z\|_2$ is given by

$$y_i^2 = \begin{cases} x_i^2 & \text{if } a_i \le x_i^2 \le b_i, \\ a_i & \text{if } x_i^2 < a_i, \\ b_i & \text{if } x_i^2 > b_i, \end{cases}$$

for $i = 1,2,\cdots,n$. Thus, $y^1 \le y^2$. Therefore,

$$f(x^1) = f(y^1) \le f(y^2) = f(x^2).$$

This completes the proof of the lemma. □

Definition 1.
For $x \in R^n$, we assign to x an integer label $l(x)$ given by $l(x) = 0$ if $x - f(x) = 0$, and

$$l(x) = \begin{cases} \min\{k \mid x_k - f_k(x) = \max_{j\in N} x_j - f_j(x)\} \\ \qquad\qquad\qquad \textit{if } x_j - f_j(x) > 0 \textit{ for some } j \in N, \\ n+1 \\ \qquad\qquad\qquad \textit{if } x - f(x) \le 0 \textit{ and } x - f(x) \ne 0. \end{cases}$$

Definition 2.

- *A q-dimensional simplex $\sigma =< y^0, y^1, \ldots, y^q >$ of $\mathscr{K}_1$ is complete if $l(y^i) \neq l(y^j)$ for $0 \leq i < j \leq q$, and $l(y^k) \neq 0$, $k = 0, 1, \ldots, q$.*
- *A q-dimensional simplex $\sigma =< y^0, y^1, \ldots, y^q >$ of $\mathscr{K}_1$ is 0-complete if $l(y^i) \neq l(y^j)$ for $0 \leq i < j \leq q$, and there is some k satisfying that $l(y^k) = 0$.*
- *A q-dimensional simplex $\sigma =< y^0, y^1, \ldots, y^q >$ of $\mathscr{K}_1$ is almost complete if labels of $q+1$ vertices of σ consist of q different nonzero integers.*

As a direct result of Definition 2, we have

Lemma 2.
Every almost complete simplex has exactly two complete facets.

Initialization: Let $K = \emptyset$, $y^0 = x^0$, $\sigma_0 =< y^0 >$, $y^+ = y^0$, and $k = 0$. Go to **Step 1**.

Step 1: Compute $l(y^+)$. If $l(y^+) = 0$, the algorithm terminates and a fixed point of f in H has been found. If $l(y^+) \in K$, let y^- be the vertex of σ_k other than y^+ and carrying integer label $l(y^+)$, and τ_{k+1} the facet of σ_k opposite to y^-, and go to **Step 2**. If $l(y^+) \notin K$, go to **Step 3**.

Step 2: If $\tau_{k+1} \subset G(\eta, K \backslash \{j\})$ for some $j \in K$, let $K = K \backslash \{j\}$ and go to **Step 4**. Otherwise, do as follows: Let σ_{k+1} be the unique simplex that is adjacent to σ_k and has τ_{k+1} as a facet, y^+ the vertex of σ_{k+1} opposite to τ_{k+1}, and $k = k+1$. Go to **Step 1**.

Step 3: If $|K| = n$, the algorithm terminates and a complete n-dimensional simplex has been generated. Otherwise, proceed as follows: Let $K = K \cup \{l(y^+)\}$ and $\tau_{k+1} = \sigma_k$. Let σ_{k+1} be the unique $|K|$-dimensional simplex in $G(\eta, K)$ having τ_{k+1} as a facet, and y^+ the vertex of σ_{k+1} opposite to τ_{k+1}. Let $k = k+1$ and go to **Step 1**.

Step 4: Let $\sigma_{k+1} = \tau_{k+1}$, y^- be the vertex of σ_{k+1} carrying integer label j, and τ_{k+2} the facet of σ_{k+1} opposite to y^-. Let $k = k+1$ and go to **Step 2**.

Theorem 2.
The algorithm terminates within a finite number of iterations with either a fixed point or a complete n-dimensional simplex.

The proof of this theorem requires the following lemma.

Lemma 3.
For any nonempty subset $K \subset N_0$ with $K \neq N_0$, there is no complete $(|K|-1)$-dimensional simplex in $G(x^0, K) \backslash H$ carrying only integer labels in K.

Proof. Let $\sigma =< y^1, y^2, \cdots, y^{|K|} >$ be an arbitrary simplex in $G(x^0, K) \backslash H$.

Consider $n+1 \notin K$. From the definition of $G(x^0, K)$, we obtain that, for some $j \in K$, $y^i_j < a_j$, $i = 1, 2, \cdots, |K|$. Thus, $y^i_j - f_j(y^i) < 0$ for $i = 1, 2, \cdots, |K|$ since $f(y) \in H$. Therefore, no vertex of σ carries integer label j. This implies σ is not a complete simplex carrying only integer labels in K.

Consider $n+1 \in K$. From the definition of $G(x^0, K)$, we obtain that, for some $j \in K$, $y^i_j < a_j$, $i = 1, 2, \cdots, |K|$, or for some $j \in N$, $y^i_j > b_j$, $i = 1, 2, \cdots, |K|$. If, for some $j \in K$, $y^i_j < a_j$, $i = 1, 2, \cdots, |K|$, then $y^i_j - f_j(y^i) \leq 0$ since $f(y) \in H$, and no vertex of σ carries integer label j. If, for some $j \in N$, $y^i_j > b_j$, $i = 1, 2, \cdots, |K|$, then $y^i_j - f_j(y^i) > 0$ since

$f(y^i) \in H$, and no vertex of σ carries integer label $n+1$. The lemma follows immediately
$\square$

Proof. of Theorem 2. Following an standard argument as that in Todd (1976), one can derive that the algorithm will never cycle. Lemma 3 implies that all the simplices generated by the algorithm must intersect with H. Since H is bounded, hence, there is a finite number of simplices intersecting with H. Therefore, the algorithm terminates within a finite number of iterations with a complete n-dimensional simplex. The theorem follows. $\square$

3 A Constructive Proof of Tarski's Fixed Point Theorem

For any given compact set C, let

$$x^{\min}(C) = (\min_{x\in C} x_1, \min_{x\in C} x_2, \cdots, \min_{x\in C} x_n)^\top$$

and

$$x^{\max}(C) = (\max_{x\in C} x_1, \max_{x\in C} x_2, \cdots, \max_{x\in C} x_n)^\top.$$

Lemma 4.
For any complete n-dimensional simplex $\sigma = < y^0, y^1, \cdots, y^n >$,

$$-\delta e \le x^{\min}(\sigma) - f(x^{\min}(\sigma)) \text{ and } x^{\max}(\sigma) - f(x^{\max}(\sigma)) \le \delta e.$$

Proof. Without loss of generality, we assume that $l(y^0) = n+1$ and $l(y^i) = i$, $i = 1, 2, \cdots, n$. Then,

$$y^0 - f(y^0) \le 0 \text{ and } y^i_i - f_i(y^i) > 0,\ i = 1, 2, \cdots, n.$$

From the definitions of $x^{\min}(\sigma)$ and $x^{\max}(\sigma)$, we know that

$$y^i \ge x^{\min}(\sigma) \text{ and } y^i \le x^{\max}(\sigma),\ i = 0, 1, \cdots, n.$$

Thus,

$$f(y^i) \ge f(x^{\min}(\sigma)) \text{ and } f(y^i) \le f(x^{\max}(\sigma)),\ i = 0, 1, \cdots, n.$$

Therefore, using $y^0 - f(y^0) \le 0$, $y^i_i - f_i(y^i) > 0$, $i = 1, 2, \cdots, n$, and $\text{mesh}(K_1) = \delta$, we obtain that, for $i = 1, 2, \cdots, n$,

$$\begin{aligned} x^{\min}_i(\sigma) - f_i(x^{\min}(\sigma)) &\ge x^{\min}_i(\sigma) - f_i(y^i) \\ &= x^{\min}_i(\sigma) - y^i_i + y^i_i - f_i(y^i) \\ &> x^{\min}_i(\sigma) - y^i_i \\ &\ge -\delta, \end{aligned}$$

and

$$\begin{aligned} x_i^{\max}(\sigma) - f_i(x^{\max}(\sigma)) &\leq x_i^{\max}(\sigma) - f_i(y^0) \\ &= x_i^{\max}(\sigma) - y_i^0 + y_i^0 - f_i(y^0) \\ &\leq x_i^{\max}(\sigma) - y_i^0 \\ &\leq \delta. \end{aligned}$$

The lemma follows immediately. □

Let δ_k, $k = 1, 2, \cdots$, be a sequence such that $\delta_k > \delta_{k+1}$ and $\delta_k \to 0$ as $k \to \infty$. For $k = 1, 2, \cdots$, let σ_k be the complete n-dimensional simplex generated by the algorithm when the mesh size of the underlying triangulation is equal to δ_k.

Theorem 3.
Every limit point of the sequence of $x^{\max}(\sigma_k)$, $k = 1, 2, \cdots$, is a fixed point of f.

Proof. Let x^* be a limit point of the sequence of $x^{\max}(\sigma_k)$, $k = 1, 2, \cdots$, and $x^{\max}(\sigma_{k_j})$, $j = 1, 2, \cdots$, be a subsequence of $x^{\max}(\sigma_k)$, $k = 1, 2, \cdots$, such that

$$\lim_{j \to \infty} x^{\max}(\sigma_{k_j}) = x^*.$$

Then, from $\|x^{\max}(\sigma_{k_j}) - x^{\min}(\sigma_{k_j})\| \leq \delta_{k_j}$, $j = 1, 2, \cdots$, we obtain that

$$\lim_{j \to \infty} x^{\min}(\sigma_{k_j}) = x^*.$$

As a result of Lemma 4, we know that

$$x^{\min}(\sigma_{k_j}) - f(x^{\min}(\sigma_{k_j})) \geq -\delta_{k_j} e \text{ and } x^{\max}(\sigma_{k_j}) - f(x^{\max}(\sigma_{k_j})) \leq \delta_{k_j} e.$$

Thus,

$$0 = \lim_{j \to \infty} -\delta_{k_j} e \leq \lim_{j \to \infty} x^{\min}(\sigma_{k_j}) - f(x^{\min}(\sigma_{k_j})) = x^* - f(x^*)$$

and

$$0 = \lim_{j \to \infty} \delta_{k_j} e \geq \lim_{j \to \infty} x^{\max}(\sigma_{k_j}) - f(x^{\max}(\sigma_{k_j})) = x^* - f(x^*).$$

Therefore,

$$x^* - f(x^*) = 0.$$

The theorem follows immediately. □

References

[1] E.L. Allgower and K. Georg (2000). Piecewise linear methods for nonlinear equations and optimization, Journal of Computational and Applied Mathematics 124, 245-261.

[2] C. Dang (1991). The D_1-triangulation of R^n for simplicial algorithms for computing solutions of nonlinear equations, Mathematics of Operations Research 16, 148-161.

[3] C. Dang (1995). Triangulations and Simplicial Methods, Lecture Notes in Mathematical Systems and Economics 421.

[4] B.C. Eaves (1972). Homotopies for the computation of fixed points, Mathematical Programming 3, 1-22.

[5] B.C. Eaves (1984). A Course in Triangulations for Solving Equations with Deformations, Lecture Notes in Economics and Mathematical Systems 234, Springer-Verlag, Berlin.

[6] B.C. Eaves and R. Saigal (1972). Homotopies for the computation of fixed points on unbounded regions, Mathematical Programming 3, 225-237.

[7] W. Forster (1995). Homotopy methods, Handbook of Global Optimization, Eds. R. Horst and P.M. Pardalos, Kluwer Academic Publishers, pp. 669-750.

[8] H. Freudenthal (1942). Simplizialzerlegungen von beschrankter flachheit, Annals of Mathematics 43, 580-582.

[9] C.B. Garcia and W.I. Zangwill (1981), Pathways to Solutions, Fixed Points, and Equilibria, Series in Computational Mathematics, Prentice-Hall, New Jersey.

[10] M. Kojima and Y. Yamamoto (1984). A unified approach to the implementation of several restart fixed point algorithms and a new variable dimension algorithm, Mathematical Programming 28, 288-328.

[11] H.W. Kuhn (1968). Simplicial approximation of fixed points, Proceedings of National Academy of Science 61, 1238-1242.

[12] G. van der Laan and A.J.J. Talman (1979). A restart algorithm for computing fixed points without an extra dimension, Mathematical Programming 17, 74-84.

[13] G. van der Laan and A.J.J. Talman (1981). A class of simplicial restart fixed point algorithms without an extra dimension, Mathematical Programming 20, 33-48.

[14] H.J. Lüthi (1975). A simplicial approximation of a solution for the nonlinear complementarity problem, Mathematical Programming 9, 278-293.

[15] O.H. Merrill (1972). Applications and Extensions of an Algorithm that Computes Fixed Points of Certain Upper Semi-Continuous Point to Set Mappings, PhD Thesis, Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, MI.

[16] H.E. Scarf (1967). The approximation of fixed points of a continuous mapping, SIAM Journal on Applied Mathematics 15, 1328-1343.

[17] H.E. Scarf (Collaboration with T. Hansen) (1973). The Computation of Economic Equilibria. Yale University Press, New Haven.

[18] M.J. Todd (1976). The Computation of Fixed Points and Applications, Lecture Notes in Economics and Mathematical Systems 124, Springer-Verlag, Berlin.

[19] A.H. Wright (1981). The octahedral algorithm, a new simplicial fixed point algorithm, Mathematical Programming 21, 47-69.

[20] Y. Yamamoto (1983). A new variable dimension algorithm for the fixed point problem, Mathematical Programming 25, 329-342.